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Teachers' perceptions of students' mathematical work while making conjectures: an examination of teacher discussions of an animated geometry classroom scenario

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Abstract

Background: This paper reports on a study examining teachers' perceptions of students they observed in an animated episode and who were engaged in the work of making conjectures in a geometry classroom. We examined eight conversations among subsets of 29 experienced geometry teachers with respect to how they described students and the mathematical work they perceived students to be engaged in.

Results: Across the study group conversations, participants described students in terms of the tasks' mathematical resources which students could understand or misunderstand and the tasks' material and social resources which they could use or misuse, but participants paid little attention to the operations that students might employ in the task or the goals that students were working toward in the task.

Conclusions: This study suggests that, when supporting students' work on conjecturing tasks, teachers focus on the tasks' resources which students use. This conjecture suggests in turn that in exchanging students' work on conjecturing tasks for claims that students have learned a bit of the geometry curriculum, teachers might deem that particular work valuable on account of the resources used.

Keywords: Geometry; Teachers' perceptions; Conjecturing; Mathematics education; Animation; Students; Task

Background

Across the world, mathematics education reform efforts and mathematics education researchers have recommended a focus on conjecturing and justification in school mathematics (Haggarty and Pepin, 2002; Jones and Fujita, 2013; Kilpatrick, Swafford, and Findell, 2001; Lannin et al., 2011; National Council of Teachers of Mathematics (NCTM), 2000; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). As reform movements - like the Common Core State Standards for Mathematics (CCSSM) in the United States - take root, teachers face the challenge of incorporating conjecturing activities into their lessons in ways that build on students' capabilities and reflect fidelity to the discipline of mathematics.

A novel feature of the CCSSM is the eight Standards for Mathematical Practice added to mathematical content standards for each grade level and domain. One of those eight Standards for Mathematical Practice, Standard 3, specifically asks students to engage in conjecturing activities: Students are to construct viable arguments and critique the reasoning of others. To meet this standard, students are asked to 'make conjectures and build a logical progression of statements to explore the truth of their conjectures' (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010, p. 6). On the one hand, the explicit push for students to engage in producing and justifying conjectures is exciting because it encourages students to engage in authentic mathematical work (as described by Lakatos, 1976; and Thurston, 1994, among others). On

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the other hand, the push could be daunting if it further complicates the work of teaching, which is probable considering the complexity of the work expected from students and the expectation that teachers be responsive to students as they support students' conjecturing.

The study reported on here explores one of the sources of increased complexity of teachers' work as they support students to engage in making conjectures: What teachers perceive in students' work as they are engaged in making conjectures. Through understanding teachers' perceptions of students while they are engaged in making conjectures, we can gain insight into the work that teachers do to support students. These insights can be instrumental in supporting teachers to expand the types of reasoning that students engage in within the classroom.

We consider this research on teachers' perceptions of students' work in context as an important complement to existing and ongoing research on teachers' perception of students' identities that pays attention to their membership in communities (e.g., Clark et al., 2009) as well as to existing and ongoing work in educational psychology on teachers' perception of students' ability (e.g., Jussim, 1989). Indeed, in a key precursor of our study, Morine-Dershimer (1978-79) noted that 'teacher conception of pupils appear to be embedded in an instructional context' (p. 43).

This paper begins with a discussion of what conjecturing activities look like in schools and the impact that teachers' perceptions of students have on students' learning opportunities. We then introduce the conceptual framework of practical rationality and show how it can be useful for understanding the work that teachers do while supporting students in making conjectures. The Results section examines the ways that groups of teachers perceived students actions as those teachers saw students engaged in the work of making conjectures. We end by sharing a conjecture about how teachers' perception of students' work in conjecturing task might inform the exchange teachers need to make of students' work on conjecturing tasks for claims that students have learned a bit of the geometry curriculum.

Reasoned conjectures in classrooms and teachers' work

Research on student thinking shows that learners are capable of making reasoned and insightful conjectures (for examples, see Balacheff, 1988; Boero et al., 1999; Ellis, 2007a; 2007b; Herbst, 2006; Lee and Sriraman, 2011; Mariotti, 2006; Martin and Pirie, 2003; Mueller and Maher, 2009; Yerushalmy, 1987, 1993). This body of research provides a template for the types of conjecturing activities that students could engage in during instruction. For instance, Ellis (2007b) shows three primary categories of conjecturing activities enacted by learners: *relating* similar situations or objects, *searching* for relationships, procedures, patterns, or solutions, and *extending* patterns or relationships into more general structures. Yerushalmy (1987, 1993) decomposed students' work on conjecturing tasks into three conjecturing activities: developing and identifying categories for generalization, forming a coherent sample of instances, and inducing generality from a sample of instances.

The examples summarized above are a small sample of the range of work that has shown learners' capabilities with respect to conjecturing and reasoning activities. They illustrate in particular that conjecturing work, when it happens, calls for students to engage in cognitive operations. However, students do not always have the opportunity to create and express these reasoned and insightful conjectures in classrooms. Writing about geometry classrooms, Herbst and colleagues have described the teacher's management of 'making conjectures' as a special case of the instructional situation of *exploration* which is a student-centered alternative for the introduction of a new idea. Herbst (2010) notes that in a situation of exploration,

> the work to be done includes the students' free choosing among a range of material operations to apply on concrete (physical or pictorial) embodiments of the concept depending the tools available to them, their reading of the particular results of those operations, and the translation of those results into general statements made in the conceptual register. The reasoning that students could thus have the opportunity to engage in can be described as abductive, proceeding from particular to general (p. 42).

To further describe what making conjectures involves in the high school geometry classroom, we propose a set of instructional norms to underpin the interaction among teacher and students around conjectures (see Table 1).

As is the case with norms for any instructional situation, the statements in Table 1 represent our hypotheses of shared expectations of the teacher and students regarding students' mathematical work and the exchange value of that work; they do not represent standards for

Table 1 Instructional norms for 'making conjectures'

Instructional norms
1. The teacher provides students with diagrams of mathematical objects and tools to use in conjecturing and prompts students to come up with conjectures
2. The teacher expects students to generate conjectures that state properties about said objects
3. The teacher does not have expectations for the generality of students' conjectures
4. The teacher enables students' to share their conjecture with the class
5. The teacher oversees discussion of each conjecture

correctness or desirability we espouse. Rather, those statements represent what we hypothesize are the norms teacher and students recognize for the situation of 'making conjectures': Our laying out of such norms is meant, like any theoretical assumption in research, as a way to simplify a system under study so that further claims about that (simplified) system can be made. This means that proposing the norms themselves is not an end in itself but a means to further understand 'making conjectures.' One could ask at some point whether those norms are indeed recognized as such by high school geometry teachers (see Herbst, et al., 2013, for an example of how to instrument such verification) but the value of asking that verification question will be heightened if one could show first that the conjectured norms are useful; this paper is concerned with those norms only insofar as their usefulness to further understand 'making conjectures.'

From the list of instructional norms in Table 1, we can imagine how the work of 'making conjectures' might unfold in classrooms. The norms suggest that instructional work includes the teacher presenting a task that provides students with mathematical, material, and social resources and asks students to make a conjecture; those resources could include a diagram and tools to gather information from it, as well as some description of the concepts students are to work with and the request to make some conjectures. Students have time to work on the task, and this work may include both observing the diagram and observing how the diagram responds to actions the students may do on them, such as drawing in or measuring attributes. The teacher expects that students will be able to craft conjectures, which will be statements about the figure represented, but more than one conjecture may be possible for students to offer for a given diagram. The teacher also does not usually expect students to have a proof for their conjecture or an explanation detailing how they arrived at their conjecture. The teacher then chooses a student to share his conjecture with the class. After the student has shared his conjecture, the teacher oversees the discussion of the conjecture, either through sharing her own comments or by asking the class to share their comments. Once the conjecture has been discussed, the class moves on to discussing another conjecture or transitions to another activity (for example, to the proof of an endorsed conjecture).

The research mentioned above suggests that 'making conjectures' unfolds in classrooms in two distinct phases: (1) students work privately (individually or in groups) making conjectures and (2) the class publicly discusses those conjectures. Herbst (2010) also notes that in an exploration,

> There are engagement stakes, according to which it is important to involve students in actively doing something self directed in geometry. There are also content stakes that include new general statements about a generic instance of an abstract concept (p. 42).

That is, *a priori*, one can expect that an exploration is done so that students consider particular ideas, but one can also expect that the exploration is done so that students participate in mathematical activity. One question to ask about situations of conjecturing is whether and how those two goals can be pursued together and how those two goals frame the way teachers perceive students' work. The current study investigates how teachers perceive students' work within 'making conjectures' and how those perceptions could impact teachers' negotiation of the stakes of the activity in terms of engagement or content. Below, we discuss research that examines the connection between teachers' perceptions of students and how those perceptions impact the opportunities that students have to engage in mathematical work.

Impact of teachers' perceptions of students on students' opportunities to learn

Research on teachers' perceptions of students has been conceptualized around what individual teachers know, believe, or notice about individual students (Hill et al., 2008; Morine-Dershimer, 1978; Sherin et al., 2010). The study reported in this paper builds on prior work to investigate the categories that the mathematics teaching profession uses to describe students, as illustrated within conversations among a group of teachers.

The categories of students that groups of teachers use to describe and discuss their students inform the ways in which teachers' actions are contingent on the work they see their students engaging in. These categories reveal clues about the features of student activity that teachers respond to and the possibilities for student action that are supported by teachers (Horn, 2005, 2007). In particular, research has shown that groups of teachers who discuss their students in dynamic and flexible ways also provide increased opportunities for students to learn (Horn, 2005). By exploring the ways in which teachers talk about students' actions while they are making conjectures, we hope to uncover the aspects of students' work that teachers respond to and the opportunities to do mathematical work that teachers create for students.

Conceptual framework

The present inquiry is inscribed within efforts to describe the practical rationality of mathematics teaching - the categories of perception and appreciation that underpin the work of teaching mathematics. Teachers' perceptions of their students are part of that rationality; those perceptions are expected to inform the decisions teachers make in instruction (Herbst and Chazan, 2011a; Green, 1976). *Practical rationality* is a general expression that covers experiences, knowledge, perceptions, and judgments that are shared among teachers of similar courses of study, such as algebra, geometry, or calculus. Practical rationality points to teaching knowledge as not limited to being rational in the sense of goal oriented or correct but also rational as sensible, adapted to the demands of the work that teachers do. This is consistent with the way in which Kahneman (2002) has described rationality as bounded to the works of two cognitive systems, a rational, explicit, and slower one, and an adaptive, tacit, and faster one. In our case, we are interested not so much on individuals' rationality but on the rationality of the professional collective for which we hypothesize some actions to be warranted on deliberate application of principles and reasons while others on the custom of replicating past patterns of interaction. This view of rationality allows for tensions, contradictions, and inconsistencies to exist in teachers' actions and decision-making and for research into that practical rationality to inquire on how teachers resolve them.

In particular, practical rationality is concerned with *categories of perception*, or how teachers perceive 'people, events, things, and ideas in the shared world of the classroom', and *categories of appreciation*, or 'the principles and qualities on which practitioners rely to establish an attitude toward people, events, things, or ideas' (Herbst and Chazan, 2011a, p. 430-1). The current study investigates teachers' categories of perception and appreciation related to students. That is, we are interested in the specific types of students that teachers perceive in the classroom and how they make sense of students' work in relation to the work of teaching.

We hypothesize that the set of categories of perception and appreciation that teachers use to describe their students is not a simple reflection of the types of students who populate their classroom. The categories reflect important features of the work of teaching and imply a social and moral order that is embedded in the work of teaching (see Bowker and Star, 1999, for a discussion of the social and moral order inherent in classification systems). By understanding the categories of perception and appreciation that teachers use in relation to their students, we learn about what teachers see as important in their work and how characteristics of student work are valued or not within the confines of that work. This in turn informs our understanding of instructional actions.

To understand classroom interactions we use a model based on the notion of a symbolic economy (Bourdieu, 1980, 1998) and a didactical contract (Brousseau, 1997). According to this model, teachers and students act as if they are managing a trade between accomplished classroom work and claims that the students have had an opportunity to learn a bit of the content at stake. The foundational hypothesis is that inside educational institutions, the teacher and her students enter into this economy because of their obligation to a didactical contract that brings students and teachers together to teach and learn mathematics. A didactical contract specifies what it means to teach and learn and what the content is that needs to be taught and learned.

Research on the use of specially designed tasks (e.g., Brousseau, 1997; Herbst, 2003) has shown that one way teachers exchange students' mathematical work for a claim on what students know is by negotiating with them how the didactical contract applies to the task when a task is enacted. This negotiation is particularly needed when the task is novel (Doyle, 1988): The negotiation may involve changes to the task itself or to what the task completion is taken to be evidence of. In the extreme, the task can be dramatically changed or its place as part of the course of studies can be severely alienated.

Another way in which the exchange between students' mathematical work and a teacher's claim on what they know is by framing such work as an instance of an *instructional situation* (Herbst, 2006; Herbst and Chazan, 2012). Instructional situations are recurrent patterns of activity that organize the actions of the students and teacher around mathematical objects. An instructional situation clusters tasks that are similar to each other in terms of what mathematical elements they contain, what actions they call forth from teacher and students, and what their completion is evidence of. In particular, tasks that are commonplace in a mathematics course, such as 'solve $2x - 1 = 3x + 4$' in algebra, do not often call for a negotiation of the task, since the word 'solve' and the existence of one variable both act as cues to conjure up what the student is supposed to do (Chazan and Lueke, 2009). In general, we hypothesize that these customary, recurrent patterns of activity, these instructional situations, make room for some canonical tasks saving teachers and students the need to negotiate how the contract applies for the task.

To conceptualize the work that students do in classrooms, we follow Doyle (1983, 1988) in using 'mathematical task' to describe self-contained segments of mathematical work that students do in classrooms. In developing his model of task, Doyle (1983) was interested in accounting for the curriculum from the students' perspective; accordingly, he described tasks as consisting of the *goal* that students are working toward, the *resources* (these could be mathematical, material, or social) that they have available to pursue those goals, the *operations* (mental or physical) that they enact to move toward their goal, and the value of the completed work with respect to an *accountability system* (particularly the grades students could get from completing the task; for an example of how students might value their work on tasks, see Aaron and Herbst, 2011). In our case, as the interest is in tasks from the teacher's perspective, the fourth element of accountability is expressed in terms of the currency the teacher manages, namely the knowledge at stake - hence the notion of instructional exchange noted above.

A goal for a task could be the construction of a conjecture. That is the case in a lesson about the angle bisectors problem, which we used to inquire on teachers' perceptions of students. The goal of the task was for students to make a conjecture about the angle bisectors of a quadrilateral: After reminding students what happens with angle bisectors of a triangle, the teacher asks 'what can one say about the angle bisectors of a quadrilateral'. In the angle bisectors problem, operations could include the drawing of quadrilaterals and their angle bisectors and observing whether the angle bisectors of those quadrilaterals meet at a point or make some other configuration. The resources could include the diagrams, prior knowledge about the definition of angle bisector, knowledge of the fact that angle bisectors of a triangle meet at a point, or time to work independently on the task.

The study reported on here informs the question of how teachers could draw on the classroom accountability system in order to exchange students' work on conjecturing tasks for claims that they have learned some bit of the geometry curriculum. We see this as a critical question related to teachers' increased responsibility to integrate conjecturing into what they are responsible for teaching. Without a clear sense of the value of students' mathematical work on conjecturing, teachers may (understandably) marginalize or exclude this work from their instruction. To inform this question, we examine how teachers perceive students' work on conjecturing tasks; such examination can provide grounds for hypotheses about how teachers could allocate value to this work. We ask the following research questions:

- What perceptions of students do teachers draw upon while observing students working independently to make conjectures?
  - What *resources* do teachers perceive students using?
  - What *operations* do teachers perceive students deploying?
  - What *goals* do teachers perceive students working toward?
- What perceptions of students do teachers draw upon while orchestrating the sharing and discussing of conjectures?
  - What *resources* do teachers perceive students using?
  - What *operations* do teachers perceive students deploying?
  - What *goals* do teachers perceive students working toward?

In thinking about teachers' perceptions of students, we hypothesize that when teachers talk about their students within the context of instruction, they describe their students in terms that can be understood to relate to the task; and that teachers might talk about their students differently depending if students are engaged in working independently to make conjectures or if students are engaged in sharing and discussing conjectures. Insights gained from these research questions will be used to inform the question of how teachers manage the exchange of students' work on conjecturing tasks for claims that students have learned a bit of the geometry curriculum. In the following section, we describe the data and analytic method that was used in the current study to inform these research questions.

Methods

The current section describes the data and analytic method used in this study. The phenomenon we are after is the collective resources that professionals who teach geometry use when observing students' work on conjecturing tasks - we elicited that data by listening to groups of teachers who were prompted to talk by a representation of instructional practice (Herbst and Chazan, 2011b) in which students were making and discussing conjectures. A representation of practice was used to create a context in which participants' discussion about students' work could be concrete. While methodologically the use of a specific classroom scenario poses questions of construct validity, we see this as a first step in exploring the sort of perceptions of students' work that teachers have; obviously, this exploration could use being followed by other studies that confront teachers with a variety of representations of students' conjecturing work.

The representation of teaching used was an animation of a classroom scenario, which had been rendered using nondescript cartoon characters with voice over. The use of an animation helped focus participants' discussion on the actions happening in the lesson (hence on the universe that contains students' mathematical work) without encouraging assumptions about the larger social context or histories of the individuals involved. Prior research (Herbst and Kosko, 2014) has shown that animations are just as good as video records in eliciting teachers' tacit knowledge of practice; we have used these representations in other studies (e.g., Herbst et al., 2011a; Weiss and Herbst, in press) where they have provided context for participants' conversations about practice and helped elicit other categories of perception and appreciation.

Our examination of group conversations among experienced geometry teachers about an animated classroom scenario used tools from systemic functional linguistics (Halliday, 1994; Martin and Rose, 2003). Our analysis identified the descriptions of students that teachers constructed during their conversations around the conjecturing task shown in the animation and resulted in a list of descriptions of students that are relevant to aspects of the conjecturing task.

Data

The data presented here were collected over the course of two school years in study groups with experienced[a] geometry teachers. Participants served students from a diverse group of schools including urban, suburban, and rural schools. Each year, two groups of five to twenty teachers met for 3 h once per month. Study group sessions around animated classroom scenarios were aimed at uncovering the practical rationality of geometry teaching; in particular, the animated classrooms scenarios were designed and employed to elicit from teachers' recognition of instructional norms that guide their work with students and mathematics content in classrooms.

In the study group sessions, participants watched and responded to animated classroom scenarios[b] in conversations with fellow participants and members of the research team. See Figure 1 for a still image from an animated classroom scenario. The animated classrooms scenarios were designed to provide enough detail about the scenario so they would evoke from teachers' relational and temporal demands of teaching while also being lean enough so they would allow for teachers to project the demands of their own teaching context (Aaron and Herbst, 2007). Participants also engaged in other activities related to the animated scenarios, like working on mathematical tasks, looking at student work, and reading and writing scripts for classroom scenarios.

The sessions were video and audio recorded and then transcribed and indexed for analysis. To index the data corpus, sessions were divided into *intervals* based on changes in the activity structure of the session (Herbst et al., 2011b; see also Lemke, 1990). An interval is a continuous length of time during a study group session in which participants are engaged in a particular activity or conversation. Herbst et al. (2011b) define it thus, '[a] n interval consists of segments of group interaction that participants construct as units of conversation by way of employing a combination of… organizational features' (p. 231). Organizational features include who the active participants are, the division of labor in the conversation, the labels that participants use to describe the theme being discussed, and length of interval (intervals are normally on the order of 2 to 8 min). The parsing of sessions according to these features resulted in intervals that cover the timeline of the session and overlap at their boundaries.

The data for this study consist of all the intervals in which the participants discussed one particular animated scenario, The Square (the plot of The Square is outlined below). The Square was watched in eight sessions, which are divided into of 368 intervals. During those sessions, The Square was discussed in 119 intervals[c]. In the remainder of the intervals, participants were discussing other animated scenarios, responding to prompts not related to any animated scenario, discussing logistics, or taking breaks.

Description of the animated scenario The Square

The animated scenario used in the study groups, The Square, shows a geometry class working on several

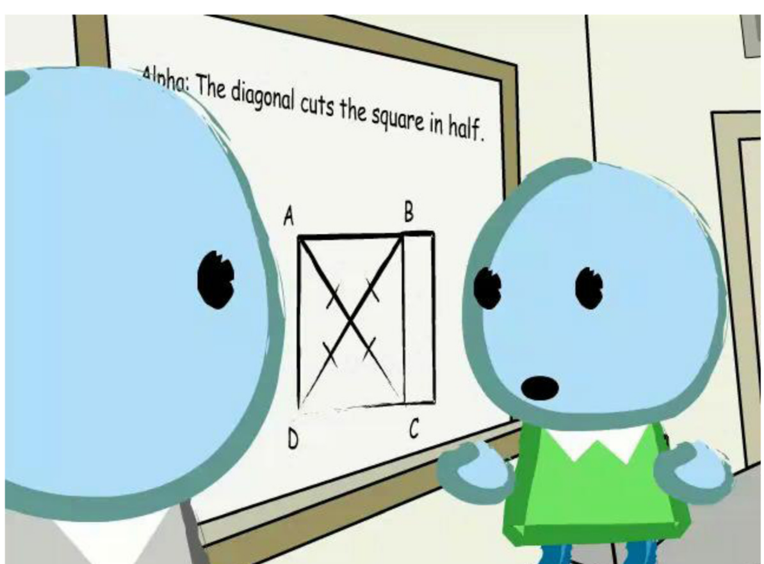

Figure 1 A student contributing to a discussion of a conjecture in the animation The Square". © The Regents of the University of Michigan, all rights reserved, used with permission.

different, but related, mathematical tasks. In this study, we examine the first few tasks, which are framed as an example of 'making conjectures.' These segments of the animated scenario elicited conversations among the participants that contained descriptions of students related to instructional work around conjecturing tasks.

The animated episode begins with the teacher giving the class the angle bisectors of a quadrilateral problem, as noted above. The teacher then says to the class 'I'll give you some time to make conjectures and then we will see if we can prove some of those conjectures'. After some time, a student, Alpha, shares a conjecture about diagonals of a square; the teacher evaluates the conjecture by reminding the class that the task is about angle bisectors; the teacher asks the class if diagonals and bisectors are the same thing; another student, Gamma, further evaluates Alpha's conjecture by illustrating that diagonals and angle bisectors are different using the case of a rectangle; and in light of Gamma's counter-example, the class reformulates Alpha's conjecture to be that the angle bisectors of a square meet at a point. The episode ends with the class attempting to prove the reformulated conjecture.

In the Results section, we share the descriptions of students that came from participants' conversations around the portions of the episode when the class was engaged in 'making conjectures'. In the following section, we describe how the data were analyzed.

Analysis of data

To analyze the data of study group conversation, we made use of systemic functional linguistics. Systemic functional linguistics (SFL) is a theory of language as a social semiotic system, developed originally by Michael Halliday (Halliday and Matthiessen, 2004). SFL looks closely at the word choices that individuals deploy in order to construct meaning through language, and it is founded on the idea that language provides organized resources to make various kinds of meaning: ideational, interpersonal, and textual. We look in particular at the ideational meta-function of language and how individuals construct the participants, processes, and context that populate discourse (Lemke, 2012). The transcripts of intervals were first coded using elements from participant analysis and cohesion chains (Halliday, 1994; Martin and Rose, 2003). Participant analysis focuses on the people and things that take part in the actions described in the conversation. Cohesion chains trace how actors and objects are represented across the conversation. These analyses provided information about how the teachers' use of words and phrases represented their perceptions of students.

Categories of perception of students were coded in the transcript whenever participants described either an animated student from The Square, a real student from their own classroom, or a hypothetical student[d] in relation to 'making conjectures'. When participants described any of these, the coding scheme recorded both the description that the participant gave and any additional information that participants gave about the student, like actions that the student performs, challenges they present for the teacher, etc. Each description was coded with respect to the dimension of the task (resource, operation, or goal) and the phase of 'making conjectures;' working independently to make conjectures, or sharing and discussing students' conjectures, that the description related to.

The methods described above allow for analysis of participants' perceptions of students and for relating descriptions of students used in discussions among teachers to how teacher perceive their students. Below, we collect the descriptions of students who teachers perceived as being relevant to the teacher's work of supporting students in making conjectures.

Results

Below, we provide examples of the ways that participants in study groups described students while discussing instructional decisions in the context of the animated classroom episode, The Square. (see Table 2 for a summary of perceptions of students.) The descriptions of students in the results are grouped by the phase of 'making of conjectures' that the described students were engaged in. We first report on how the students were

Table 2 Focal students while the teacher facilitates work on conjecturing tasks

Phase of activity	*Focal students*	*# Descriptions across # intervals*
Making conjectures	• Students who misuse task resources	10/7
	• Students who would benefit from the use of additional resources	8/6
	• Students who might encounter difficulty making a conjecture	21/16
Sharing and discussing conjectures	• Students who share their conjecture at the board	5/4
	• Students who have trouble making sense of their peers' conjectures	11/7
	• Students who used their peers' conjectures to better understand their own conjecture	3/3

described with respect to working independently to make conjectures; then we report on how students were described with respect to sharing and discussing conjectures. Over the 119 intervals that were coded for perceptions of students, 58 descriptions of students emerged across 36 intervals. Thirty-nine descriptions were related to working independently to make a conjecture, and 19 descriptions were related to sharing and discussing a conjecture.

Perceptions of students related to working independently to make a conjecture

The animation The Square gave participants a context to discuss students' independent work making a conjecture. While students are working on crafting a conjecture, the teacher has the opportunity to guide and observe each dimension of the task; namely, the resources that students use, the operations that students perform, and the goals that students work toward. Below are descriptions of students from the participants' discussion of how students work on the conjecturing task. In the data, we see that the majority of participants' descriptions of students related to working on the task can be placed into three categories: students who misuse task resources, students who would benefit from the use of additional resources, and students who might be unsuccessful in making a conjecture.

Students who misuse task resources

When participants described students working on the conjecturing task, we see evidence that they perceived students in light of students' misuse of task resources, in particular, angle bisectors and diagonals. Students were described as misusing task resources ten times across seven intervals. One participant, Tina, expected that students' confusion around angle bisectors and diagonals was inevitable. She said, 'I think that confusion would've been there, no matter what. When you say angle bisectors, half the class is going to [think] diagonals anyway' (TMW111506, 40, 1115). Another participant, Denise, said that if she insisted to her students that diagonals and angle bisectors are different, her students would look at their diagram of a square and respond, 'No, you don't know what you're talking about because they are the same' (TMW111506, 48, 1326). In the quotes, we see the participants anticipating that students will conflate the resources of the task.

Students who would benefit from the use of additional resources

We also see evidence in the data to support the claim that the participants perceived students in terms of additional task resources that would improve students' success with making conjectures. Across six intervals, participants provided eight descriptions of students related to additional resources aimed at improving students' work on making conjectures. Participants saw that the animated teacher could have provided the class with a worksheet that contained several examples of one type of quadrilateral, several different quadrilaterals, or a hierarchical list of the quadrilaterals (Jillian, TMW111506, 30, 830; Jillian, TMW111506, 30, 832; James, ThEMaT081905, 9, 142). Raina suggested that students could have worked with a partner. She said, 'maybe [Alpha's] got the kid that he's sitting next to that he's working with and they come up with this idea together' (TMT110706, 17, 206). Lucille recommended that students could have use technological support. She said, 'I think some kids like the computers or the calculators' (ESP091305, 4, 45). Thus, additional resources like worksheets that structured students' work, provided diagrams of quadrilaterals, partners for students to share their thinking with, or extra tools were suggested as things that students could use.

Students who might encounter difficulty making a conjecture

In the study group data, participants described students who they expected might encounter difficulty as the class worked toward making conjectures. In 16 intervals, there are 21 instances of participants describing how students might be unsuccessful in arriving at a conjecture. Tina said, 'I'll have kids who'll draw a square three times in a row. "Well, draw something different than a square"… Other kids who have done, maybe done three different ones, you might just say, "Good job", you know, "Keep going" you know, "Draw some conclusions"' (TMW111506, 13, 359). Tabitha indicated which students would have needed help, 'I'd probably start with the kids who are sitting there, either talking to their neighbor or staring at the wall and say, "alright, well, draw something with four sides. Draw in angle bisectors. Draw something else with four sides"' (TMW111506, 12, 327). Participants perceived that students might be unsuccessful for a variety of reasons, including oversimplifying the problem by not making use of the range of useful resources or operations or not being able to formulate a plan or enter into the work.

Looking across these categories of students that participants described with respect to working independently to make conjectures, we see that participants are primarily concerned with ensuring that students understand the resources of the task, such as diagonals and angle bisectors, and that students will know what operations they can deploy with those resources. We see evidence that participants perceive students in terms of their successful use of appropriate resources and operations, but we do not see evidence that participants attend to the goal of the task, the conjecture that students are working toward. In particular, students were not differentiated in terms of what conjecture they could come up with. Rather, participants appeared to accept any conjecture students might come up with as long as resources had been used appropriately.

Perceptions of students related to sharing and discussing students' conjectures

The animation The Square also gave students a context to respond to how a class might discuss students' conjectures. While students are sharing and discussing their conjectures, the teacher has the opportunity to choose which students share conjectures, which conjectures get shared, and how students interact with each other's conjectures. Below are descriptions of students from the participants' comments on how students share and discuss conjectures. In the data, we see that the majority of participants' descriptions of students related to sharing and discussing conjectures can be placed into three categories: students who share their conjecture with the class, students who have trouble making sense of their peer's conjecture, and students who use their peers' conjectures to better understand their own conjecture.

Students who share their conjecture at the board

Participants' discussed students sharing their conjectures using only five descriptions across four intervals. Despite the critical impact that the choice of which student shares and which conjecture is shared can have on the class' discussion, we do not have much evidence about how participants perceive students who are engaged in this work. Megan said that she would call on the student who explored an interesting case, 'The minute I was done with [Alpha's conjecture], I would've picked [the student with a conjecture about a kite]' (TMT110706, 13, 123). Denise saw the reasonableness of recording several conjectures with the hope that collectively they would lead to an interesting conclusion, 'I think they were trying to just put all the points up there so you can come up with one big point' (TMW111506, 37, 985). Across the data, participants described students in relation to the quadrilateral that they used as a basis for their conjecture and in relation to the number of students who would have their conjecture shared. One could imagine that participants would describe students in terms of the content of their conjecture or the process that students used to arrive at their conjecture; however, this is not apparent in our data.

Students who have trouble making sense of their peer's conjecture

The majority of participants' descriptions of students related to sharing conjectures reflect a concern that students in the class will have difficulty making sense of the conjecture that is shared. Participants describe students as having difficulty making sense of conjectures shared by others in 11 descriptions across seven intervals. As students shared conjectures that distinguished between diagonals and angle bisectors, Tina said that she would work to emphasize the distinction; she said, 'Because half the class is going be thinking "diagonal"' (TMW111506, 43, 1201). Melanie worried that students would have difficulty making sense of complicated diagrams that accompanied conjectures. She said, 'There's a few [students] that would understand [the diagram] but it would lose the majority [of students]' (TMT110706, 55, 767). Some of participants' descriptions of students revisit the issue of misusing task resources, noting that confusion could prevent students from understanding the conjectures shared by their peers. Participants also describe students in relation to confusion that students might experience when confronted with a complicated diagram accompanying another student's conjecture. Both of these barriers to making sense of classmates' conjectures are tied to the resources that are used as the foundation of the conjectures being shared.

Students who used their peers' conjectures to better understand their own conjecture

In participants' descriptions of students related to sharing and discussing conjectures, we see evidence of the perception that students' discussion of their peers' conjectures can improve their understanding of their own conjectures. We see three descriptions of this type across three intervals. In response to a student who overgeneralized from the case of a square, Tina planned to ask a second student who made a conjecture about rectangles to share their conjecture to contradict the first student's claim. She said, 'And hopefully you'll have the kid who draws the rectangle [share their conjecture]' (TMW111506, 14, 395). We see evidence that participants expect that students could conclude that their conjecture was incorrect based on a peer's conjecture. The sharing and discussing of conjectures could support students in revising their conjectures without direct intervention from the teacher.

Looking across the three categories of students that participants describe with respect to sharing and discussing conjectures, we see evidence that participants attend to which students share their conjectures, how students are making sense of the conjectures shared by their classmates, and how students deepen their understanding of their own conjecture through discussing their classmates' conjectures. Interestingly, we again see participants describing more students in terms of the resources that they use to make conjectures and less description of students in terms of the conjectures that are shared or students' strategies for making conjectures.

The participants' comments highlighted above inform the question of how teachers perceive students as they are engaged in making conjectures. The descriptions of students were collected from conversations around the animated story, The Square; however, the ways in which participants described students are arguably relevant to teachers' work of supporting students in making conjectures in other geometric contexts. The comments that the animated episode elicits provide more than just information about participants' perception of the scenario presented in The Square; they can also be used to gain a deeper understanding of the categories of perception that guide teachers' work in classrooms. In the following section, we discuss the observations in terms of how teachers' perceptions of students shape students' work on making conjectures.

Discussion

From examining teachers' descriptions of students, we learn about the specific ways that students' work on conjecturing tasks is perceived by teachers and gain insight into how teachers manage the exchange of students' work on conjecturing tasks for claims that students have learned a bit of the geometry curriculum. In response to the first phase of work, working independently to make conjectures, teachers were concerned with the task resources that students use to make conjectures - in particular, teachers were concerned with how students misused task resources and what additional resources they might benefit from using. Additionally, teachers were concerned with the difficulties that students might encounter while working to make conjectures, such as not using the full range of resources and operations that they had available to them. In response to the second phase of work, sharing and discussing conjectures, we see teachers still attending to the resources that students are using (or misusing) with some attention to the operations that students perform. We see this through how teachers frame their concern with which students would share conjectures, how students would make sense of each other's conjectures, and how students would better understand their conjecture through discussing other students' conjectures.

As we look across teachers' descriptions of students with respect to the conjecturing task, it is striking that teachers describe students primarily in terms of the task resources, occasionally in terms of the task operations, and descriptions of students in terms of the task goal are absent. One can contrast that perception of students from what one might expect from other stakeholders. From the point of view of a mathematician, for example, the most important aspect of a conjecturing task might be the goal, the conjecture that results from the conjecturing activity, and how that conjecture expands the sphere of potential mathematical knowledge (Thurston, 1994). One could imagine that a mathematician would attend to the conjecture that students produced and how they worked toward or away from particular conjectures. In contrast, mathematics educators and learning scientists are likely to be cued into the task operations that students make use of when engaged in a task. Looking at the research on students' conjecturing, we see mathematics educators attending to the particular moves that students make while they are conjecturing, such as; relating similar situations or objects, searching for relationships, procedures, patterns, solutions, and extending patterns or relationships into more general structures (Ellis, 2007b) or developing and identifying categories for generalization, forming a coherent sample of instances and inducing generality from a sample of instances (Yerushalmy, 1987, 1993), and how these moves support students in successfully making conjectures.

Teachers' attention to the resources that students' use on conjecturing tasks may seem unlikely given what one might expect from mathematicians or mathematics educators; however, we see this attention consistent with teachers' obligations to value students' work in terms of the geometry curriculum. A principal component of the curriculum in high school geometry is knowledge of the mathematical concepts that students would be making conjectures about; quadrilaterals, angle bisectors, diagonals, etc. The specific operations that students perform during conjecturing tasks are not part of what the teacher is responsible for within the high school geometry curriculum. In terms of operations, teachers are responsible for supporting students in learning much broader categories of work, such as 'logical reasoning' or 'critical thinking.' Teachers hold themselves accountable to teach these broader operations during situations of 'doing proofs' when students are expected to work on proof tasks (Herbst and Brach, 2006). Within the 'doing proofs' situation students' work is exchanged for the claim that students have developed skill with specific operations, such as applying theorems and identifying giving information.

Herbst (2010) had described and illustrated how situations of exploration can be used to engage students in the introduction of new material. Clearly, conjecturing tasks could usher a class into a new theorem and some curricula have been developed to introduce geometric theorems through conjecturing (Serra, 1997). But new theorems are not necessarily introduced through engaging students in conjecturing tasks; quite often, they are merely installed by the teacher (Herbst et al., 2011b). Situations of geometric calculation (Hsu and Silver, 2014) or of doing proof (Herbst and Brach, 2006) are often used to create context for students to show they know those theorems. Situations of exploration seem however useful for teachers to ensure student participation and engagement; our data do show their concern with enabling and sustaining this participation.

Despite appearing as if teachers are not attending to important aspects of students' work on conjecturing tasks, we see that teachers' attention can be attributed to their responsibility in the didactical contract and the need to manage the exchange between students' work on conjecturing tasks and claims that some piece of the geometry curriculum has been taught and learned. We see evidence that teachers are attuned to the resources that students make use of while working on conjecturing tasks, and we can see that knowledge of these resources, such as mathematical objects and their definitions, are principal components of the geometry curriculum that high school geometry teachers see as exchanging for students' work on conjecturing tasks.

Conclusions

In this paper, we report on a study that looked at geometry teachers' perceptions of students while they are making conjectures. Scholars in educational psychology and in the growing area of mathematical identity have described students and teachers' perceptions of students in terms of enduring or emerging personal characteristics. For example, see work on students' open and closed understandings of mathematics (Boaler, 1998; 2002), growth and fixed mindsets (Dweck, 2006), grit (Duckworth et al., 2007; Tough, 2013), and classic work on multiple intelligences (Gardner, 1993). In contrast with those studies and following our earlier work (Aaron and Herbst, 2012) looking at students' in the context of the mathematical work they do, this study looks at teachers' perceptions of students within work environments, particularly within making conjectures. The generalizability of our findings is limited by the fact that we only looked at making conjectures, we relied on responses from a small number of practitioners to a particular scenario, and we analyzed these data qualitatively. In spite of those limitations, the findings suggest that in the context of conjecturing tasks, teachers perceive students primarily in terms of the resources that students use (and misuse) and in terms of their level of engagement. These perceptions can be attributed to the didactical contract of the high school geometry classroom and the pieces of the curriculum that teachers see as corresponding to students' work on conjecturing tasks.

As teachers frame conjecturing tasks as primarily about the resources that students use, we see that the operations that students use to make conjectures and the conjectures that students arrive at could be underspecified and undervalued in classrooms. As a result, 'making conjectures' could be seen as an ancillary activity. One way to increase the specificity and value of students' work on conjecturing tasks would be to increase and broaden the role that conjecturing plays in complying with the didactical contract in geometry. This might begin with policy changes, such as the inclusion of Standards for Mathematical Practice in the Common Core State Standards for mathematics. Further, conjecturing, including the specific operations that students engage in and the conjectures that they produce, would need to appear prominently in artifacts that shape teachers' vision of the curriculum, including textbooks and local and national assessments. This study speaks to the interplay between what students know and are able to do mathematically in classrooms and the impact that teachers' perceptions of students' work have on the mathematical opportunities that students have in classrooms.

Endnotes

[a]We considered teachers to be experienced if they had taught geometry for at least 3 years.

[b]Animated classroom scenarios created by this project, including the animated scenario used in this study, can be viewed online at www.lessonsketch.org.

[c]When quotes from the transcripts of sessions are displayed in the Results section, they are labeled with a parenthetical citation that contains the name of the session, the interval number within the session, and the transcript line number within the session. Session names identify each of the groups and preserve the dates when the sessions were held.

[d]Herbst (1998) discusses the *modal student* or 'a hypothetical person playing the role [of student] generalized across all students in the class' (p. 150).

Abbreviations

CCSSM: Core State Standards for Mathematics; NCTM: National Council of Teachers of Mathematics.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

WA contributed to the design of the animation, proposed this research project, performed the analysis of the discourse data, and drafted the manuscript. PH was the main designer of the animations and of the larger study to which this project contributed; PH contributed to the framing and final writing of the article. All authors read and approved the final manuscript.

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