Altindis *et al.*
International Journal of STEM Education (2024) 11:32
https://doi.org/10.1186/s40594-024-00492-5

International Journal of STEM Education

RESEARCH Open Access

Exploring the role of disciplinary knowledge in students' covariational reasoning during graphical interpretation

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Abstract

Background This study investigates undergraduate STEM students' interpretation of quantities and quantitative relationships on graphical representations in biology (population growth) and chemistry (titration) contexts. Interviews ($n = 15$) were conducted to explore the interplay between students' covariational reasoning skills and their use of disciplinary knowledge to form mental images during graphical interpretation.

Results Our findings suggest that disciplinary knowledge plays an important role in students' ability to interpret scientific graphs. Interviews revealed that using disciplinary knowledge to form mental images of represented quantities may enhance students' covariational reasoning abilities, while lacking it may hinder more sophisticated covariational reasoning. Detailed descriptions of four students representing contrasting cases are analyzed, showing how mental imagery supports richer graphic sense-making.

Conclusions In the cases examined here, students who have a deep understanding of the disciplinary concepts behind the graphs are better able to make accurate interpretations and predictions. These findings have implications for science education, as they suggest instructors should focus on helping students to develop a deep understanding of disciplinary knowledge in order to improve their ability to interpret scientific graphs.

Introduction

Graphical interpretation skills are foundational for communicating ideas, analyzing information, and making well-founded scientific judgments (Glazer, 2011). Across the disciplines in STEM (Science, Technology, Engineering, and Mathematics), students struggle when interpreting graphs of continuous, covarying quantities (Altindis, 2021; Carlson et al., 2002; Moore, 2014; Moore & Thompson, 2015; Weber, 2012). Interpretation in these cases relies on covariational reasoning: the ability to understand and analyze the relationship between two variables that change simultaneously (Carlson et al., 2002). Covariational reasoning is cognitively demanding, requiring students to imagine measurable attributes, conceive of problems or contexts related to those quantities (i.e., quantitative reasoning; Thompson, 1993), and comprehend the emerging dynamic quantitative relationship(s) depicted by the graph, such as whether the coordinated changes among variables exhibit curvature or follow straight lines (Altindis, 2021; Carlson et al., 2002; Oehrtman et al., 2008).

In scientific fields, graphed variables are related to each other by fundamental principles grounded in disciplinary knowledge. This suggests that one's ability to understand how the principles connect to the variables might aid in interpreting the correlated values and changes in the variables. In other words, science disciplinary knowledge

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may support covariational reasoning. Science disciplinary knowledge may help students form mental images of the variables, which Thompson (1993) argues is critical to covariational reasoning. This study explores the covariational reasoning ability of college students and the extent to which their science disciplinary knowledge (SDK) in biology and chemistry may support mental imagery. By exploring this process, we hope to provide deeper insights into how to support students in developing robust covariational reasoning abilities as they engage in scientific graphical interpretation.

Literature review

Graphical representations (hereafter 'graphs') are essential for summarizing scientific data and presenting relationships between variables (Latour, 1987; Lemke, 1998; Roth et al., 1997). Therefore, a critical learning objective for STEM students is to develop graphical interpretation skills (e.g., Qiao et al., 2024). We define "*graphical interpretation*" as the cognitive processes and intentional actions undertaken by students to discern information and construct understanding from a graph. Graphical interpretation relies on a hierarchical set of steps, from elementary to advanced comprehension levels: (1) identify important visual features (e.g., axis titles, data points); (2) understand the trend or relationship represented by the data; and (3) make disciplinary inferences or predictions based on the data (Curcio, 1987; Glazer, 2011). Supporting this perspective, eye-tracking studies have found that novices spend more time on informational components of graphs (title, axes) and describing general trends, whereas individuals with greater expertise spend more time analyzing and interpreting data in describing specific trends (Atkins & McNeal, 2018; Harsh et al., 2019; Ruf et al., 2023).

Several categories of graphical misinterpretations of covarying quantities have been identified (Carlson, 1998; Carlson et al., 2002; Monk, 1992). One common misinterpretation is confusing graph height with line slope (Leinhardt et al., 1990), observed in physics (Ivanjek et al., 2016; McDermott, 1987) and chemistry (Heckler, 2011). Another difficulty is the relationship between slope of the line and rate of change (e.g., Bowen et al., 1999). When graphs represent non-linear (curved) relationships, it can be particularly challenging for students to interpret changes in slope and, therefore, to make correct inferences about changes in the rate of change (McDermott et al., 1987; Planinic et al., 2013). Furthermore, when interpreting graphs of dynamic phenomena, students commonly read the x-axis incorrectly as time (e.g., Atkinson et al., 2021; Jones, 2019; Popova & Bretz, 2018), which would affect predictions or inferences.

Another challenge in graphical interpretation rests in how students perceive or encode the shape of the graph in ways that divert attention away from the variables and their quantitative relationship (Moore & Thompson, 2015; Saldanha & Thompson, 1998). Some students may view the graph as an iconic image, or literal physical representation of the phenomenon being depicted (Carlson, 1998; Leinhardt et al., 1990; Monk, 1992). This commonly occurs with graphs depicting distance or position with respect to time, where students interpret the lines as the actual paths of people or objects (e.g., Lai et al., 2016; McDermott, 1987). Students may also employ static shape thinking, where they view the graph as a shape or a malleable "piece of wire" (Moore & Thompson, 2015). A series of articles in chemistry (Parobek et al., 2021; Rodriguez & Towns, 2019; Rodriguez et al., 2018, 2019) and mathematics (Rodriguez & Jones, 2024) proposed the idea of making sense using 'graphical forms'—intuitive ideas based on perceived graphical features—such as 'steepness as rate' and 'straight means constant'. Although use of graph shape as a heuristic can be productive in interpreting some aspects of scientific graphs or representations (Talanquer, 2022), it can limit students' understanding of the represented phenomenon as a dynamic process (Rodriguez et al., 2019).

Covariational reasoning

Covariational reasoning is the complex cognitive process of coordinating two changing quantities—how each quantity changes in tandem with each other (Carlson et al., 2002). The construct developed out of Thompson's (1993) theory of quantitative reasoning, where quantities are defined as measurable attributes of an object. Thompson situated his theory on Piaget's (2001) work on the mental images students create. Mental imagery involves a re-presentation of an experience by recalling and reconstructing previous encounters (Thompson et al., 2024; Von Glasersfeld, 1991). These mental images are not limited to visual images, but encompass all types of mental representations that represent past experiences, thoughts, emotions, and sensations related to the quantities (Thompson et al., 2024). Creating mental images is cognitively demanding, requiring the conceptualization of quantities (quantity in mind, not the real world), quantification (an act of conceiving an object and assigning a unit of measure to one of its attributes), and relationships among quantities (Thompson, 2011). When reasoning about covarying quantities, Thompson (1994) asserted that students needed to form a series of mental images: an image of the change in one quantity, then an image of the coordinated change between two quantities, and finally an image of the two changing quantities covarying simultaneously. Creating this series of mental images is essential for engaging in covariational reasoning.

Mental images serve as the basis for the mental actions that underlie Carlson et al. (2002) covariational reasoning framework. At the lowest level of mental action (MA1), students broadly coordinate change in one variable with change in the other variable, verbalizing for instance that the dependent variable changes as the independent variable changes. Moving up the levels, students verbalize the direction of change (MA2), how much the dependent variable changes with consistent increases in the independent variable (MA3), and the average rate of change for consistent increases in the independent variable (MA4). At the highest level (MA5), students understand how the instantaneous rate of change changes for the function with respect to a consistent change in the independent variable.

Disciplinary knowledge in graphical interpretation

Prior knowledge related to graph content can affect graphical interpretation, hindering or helping (Atit et al., 2020; Glazer, 2011; Kozma & Russell, 1997; Roth & Bowen, 1999, 2001; Roth & McGinn, 1998; Schönborn & Anderson, 2009; Shah & Hoeffner, 2002). Prior knowledge has been shown to sway viewers toward making incorrect inferences from graphs rather than using the evidence depicted in the graph. For example, college students used their knowledge of drunk driving and car accidents to infer a causal relationship on a graph that depicted no such relationship (Shah & Hoeffner, 2002). Alternatively, prior knowledge can also support the interpretation of familiar data, allowing viewers to make more inferences and provide deeper explanations (Glazer, 2011; Shah & Freedman, 2011). For example, students across different educational levels, from high school to graduate school, demonstrated better performance in graphical interpretation when they had more extensive prior knowledge of the context and weaker performance where they lacked that familiarity (Aoyama, 2007). More recently, Edelsbrunner et al. (2023) found a positive relationship between representational competence and conceptual knowledge in undergraduate physics students.

Relevant prior knowledge includes disciplinary concepts: core principles, explanations, and theoretical constructs (NGSS Lead States, 2013), as well as knowledge of relevant contexts, exemplars, and priorities for a given discipline—all of which serve as resources for sense-making (e.g., Bowen et al., 1999). This knowledge develops over time through experience in a discipline, meaning it is content-specific and advanced rather than intuitive or naive, so an individual would be unlikely to use it without formal disciplinary experience (diSessa, 1993, 2018; Richards et al., 2020). An individual with more disciplinary expertise is expected to have a body of well-connected, detailed knowledge resources organized around core concepts that guides and supports their use (e.g., Chi et al., 1981; Larkin et al., 1980; NRC, 2000). For example, work in biology has demonstrated that scientific experts with relevant disciplinary knowledge are able to connect a graph to situations they have experienced or common practices in the field, which allows them to draw deeper inferences during graphical interpretation (Bowen et al., 1999; Roth & Bowen, 2001, 2003). By comparison, university students with limited or ambiguous disciplinary knowledge have more difficulty deriving meaning (Bowen et al., 1999). Other studies have demonstrated that students with greater knowledge of a disciplinary concept (e.g., cellular transport) are able to use that knowledge to make connections between molecular and macroscopic representations and do not focus on surface features that distract students with less knowledge (Cook et al., 2008). Disciplinary knowledge, therefore, serves as a reference that students can productively use when interpreting graphs (e.g., Berg & Moon, 2023).

Covariational reasoning in scientific graph interpretation

A few recent studies suggest that understanding the physical phenomenon, its quantities, and related disciplinary concepts, affects interpretation of graphs representing covariational relationships. The implication is that covariational reasoning is different in science and that covariational reasoning frameworks from mathematics education may fail to fully describe it (e.g., Olsho et al., 2022; Zimmerman et al., 2019, 2023). For example, in physics, fourth-year students were better able to answer questions about slope situated in physics graphs than in finance graphs (Susac et al., 2018). This could be due to the integration of disciplinary knowledge during problem-solving, and work in physics suggests that richer responses are produced when students scaffold disciplinary reasoning with mathematical reasoning about quantity (Brahmia, 2019). Comparatively, work in biology (Covitt et al., 2021; Reichert et al., 2015; Sterman & Sweeney, 2007) has indicated that when individuals interpret representations of phenomena involving matter flow and accumulation, they often depend on basic approaches like heuristic reasoning or pattern-matching and show limited application of disciplinary knowledge instead of more sophisticated covariational reasoning. Scott et al., (2023), who examined student reasoning about mass balance in biological systems, reported that to engage in more sophisticated covariational reasoning, students had to establish a relationship between the rate of change, which describes the overall flow, and a single-unit variable, such as an amount of matter. That is, engaging disciplinary knowledge of mass balance facilitated covariational reasoning.

Still, these analyses fail to fully acknowledge the importance of quantitative reasoning theory and the use of mental imagery, which are essential for covariational reasoning. This study intended to explore the interrelationships among science disciplinary knowledge, mental imagery of variables, and covariational reasoning via the following research questions:

RQ1. To what extent do students engage scientific disciplinary knowledge when interpreting scientific graphs of covarying quantities?
RQ2. In what ways does mental imagery grounded in scientific disciplinary knowledge play a role in students' covariational reasoning?

Methodology

Setting and population

This study was conducted at a research-intensive university in the northeastern United States. The student population is primarily White (82%) and has more students identifying as female (56%) than male (44%). Participants were recruited from large enrollment (100 plus) introductory courses: two sections of biology, two sections of chemistry required of life science majors, and one section of chemistry for engineering majors. The study was approved by the study institution's IRB (IRB-FY2021-69). Students selected for interviews were identified according to performance on a set of 20 survey questions (forced choice and open-ended) designed to assess skills in proportional and covariational reasoning and in graphical interpretation. Detailed analysis of the results of this assessment is not pertinent to the current study and therefore not discussed here. Students were stratified by percent accuracy scores into low (score 0–33%), middle (34–66%), and high (67–100%) performance categories. Four students were recruited from each stratum ($N=16$ total) for hour long semi-structured interviews (Goldin, 2000). Interviews were conducted within the first month of the semester and were repeated with the same students (less one, $N=15$) in the last several weeks of the semester.

Interview protocol and tasks

A semi-structured interview protocol, developed by the math expert and then refined and revised by the science experts, was employed (protocol in Supplementary Information). The math expert (NA) conducted and video-recorded all interviews. Interview participants completed three graphical reasoning tasks that spanned chemistry, biology, and mathematics contexts. The data presented here relate to two tasks based on the same graph shape (S-shape) but in different disciplinary contexts: in biology, as a population growth curve for brown tree snakes (from the post-interview); and in chemistry, as a strong acid–strong base titration (from the pre-interview). All students were given the titration task ($N=15$), while only students enrolled in the introductory biology class were assigned the population growth task ($N=8$). Students were given 5–10 min to work on each task and then they were encouraged to verbalize their thoughts and problem-solving processes. Interviews were transcribed verbatim from the original video using automated speech recognition software and then edited by one or more of the authors. Images of written artifacts and descriptions of student gestures were inserted, where appropriate, to generate enhanced transcriptions.

The multidisciplinary team composed of biology (BC and MLA), chemistry (KAB and CFB), and mathematics (NA) experts collaborated to analyze the data. We recognize that our experiences within these disciplines influenced our interpretations of student responses during data analysis, especially when interpreting students' SDK. To help with this, we entered these tasks with the understanding that students will utilize language that resonates with them and tried to maintain the students' voices. During interviews, students could freely discuss what they felt was relevant to the questions. We also used student responses to guide our categorization of basic, mid, and high SDK in an attempt to diminish our expert bias towards particular connections or terms. However, it is important to recognize and define our role as outsiders in this research context (Narayan, 1993), given that we have a limited understanding of participants' individual differences and identities when designing and conducting this study.

Biology task: population growth

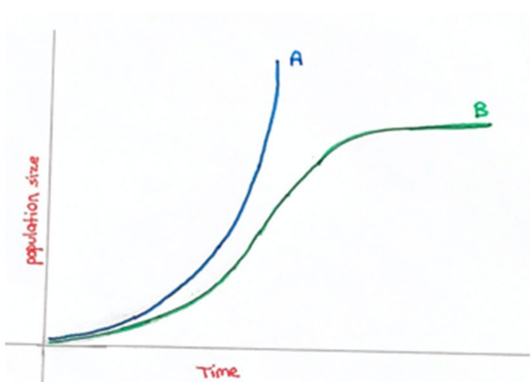
In biology, population growth models, which are based on population birth and death rates, are used to make predictions about a population's trajectory. When birth rates are higher than death rates, populations grow, whereas when death rates are higher than birth rates, populations decline. The logistic population growth model applies to a habitat where there is competition for resources (food, water, space) within a species. Populations initially increase exponentially, but as resources become more limited due to the increase in the number of individuals in the population, population growth slows down (birth rates decline and/or death rates increase). The point at which resources start to limit growth is the inflection point on the logistic graph. Eventually, the population reaches a maximum size that the environment can support, which is called the carrying capacity. At carrying capacity, limited resources constrain per capita birth and death rates to be equal, resulting in no change in population size over time.

The biology population growth task was created by the authors to showcase both an exponential growth and logistic growth model and included a link to a video showing how the population sizes change over time. After watching the video, participants were asked to describe the relationships depicted in each graph and prompted to compare the exponential and logistic graphs. Additionally, participants were asked to sketch graphs showing how population growth rate changes over time for each model (Fig. 1).

Chemistry task: titration experiment

In chemistry, titrations are used to determine the concentration of one substance (the analyte) by slowly adding to it another substance (the titrant) that it reacts with. Because each drop of added titrant reacts with and removes a fixed quantity of the analyte, the proportional effect on analyte concentration is small at the start but increases continuously. When the analyte is nearly exhausted, the fractional change is orders of magnitude larger. The amount of titrant added to this point then indicates how much analyte was present at the start, a useful piece of chemical information. In this example, the analyte is an acid, and the titrant is a base. They react to form water, which itself contributes minimally to the measured acidity of the solution. The Y axis is typically graphed as the logarithm of the analyte concentration in units of pH (pH = − log[acid concentration]) versus number of drops or cumulative volume of added base. So, the point at which the acid analyte is exhausted appears as a steep change of pH and inflection in the graph. Before inflection, the base addition causes an increasing pH changing at an increasing rate; after the inflection, the pH increases at a decreasing rate as the base added with each

This graph and video represent population size and time. For example, on the graph, A represents the population growth of Burmese pythons in South Florida, B represents the population growth of brown tree snakes in Guam.



1) How does the population growth of Burmese pythons in South Florida and the population growth of brown tree snakes in Guam change with time?
I think…

2) What do you notice about differences in the population growth of Burmese pythons in South Florida, the population growth of brown tree snakes in Guam?
I think…

3) Sketch a graph showing population growth rate versus time for Burmese pythons in South Florida.
a. Explain how the population growth rate and time are changing on your graph.

4) Sketch a graph showing population growth rate versus time for the brown tree snakes in Guam.
a. Explain how the population growth rate and time are changing on your graph.

Fig. 1 Population growth (biology task). Students were also provided with a simulation showing the production of the lines on the population graph over time (see Supplementary Information) drop causes a smaller and smaller effect, approaching the pH of the added base solution. Because of chemical equilibrium principles, the pH of the solution after inflection is controlled by the amount of base present; more base forces a larger pH value. Unlike the biology graph, which emerges because of the comparative rates of births and deaths affecting the number of organisms at any time point, the chemistry graph emerges because of logarithmic transformation of the fractional changes in concentration of one chemical entity as it disappears and a new chemical entity as it appears over time.

After reading the task prompt, each participant watched a video showing production of a titration curve during a titration experiment (see Supplementary Information). Participants were then asked to explain the relationship between the pH of the solution and the amount of base added (Fig. 2). Probing questions were asked with the intent to explore whether participants could differentiate between the nature of the curvature (concave up and concave down) around the inflection point and understand how such a graph represents the covarying quantitative relations between the volume of base added and measured pH.

Analysis

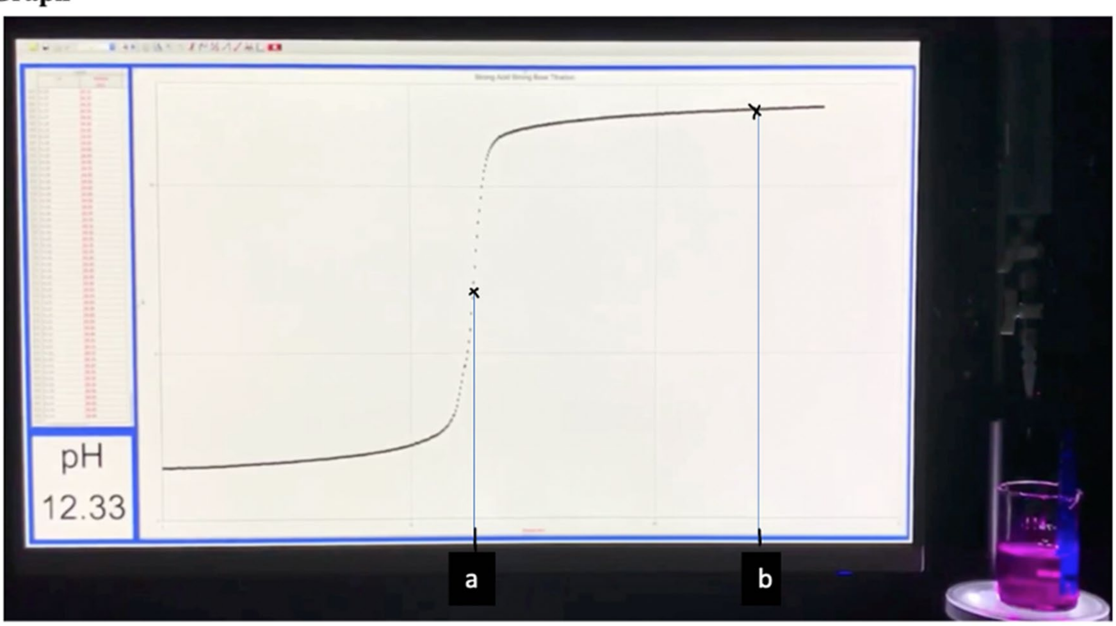
Deductive coding with covariational reasoning framework

To characterize students' covariational reasoning, the five-level framework developed by Carlson et al. (2002) was used. To establish an understanding of what covariational reasoning looked like in each task context, the team articulated the mental actions from each disciplinary perspective. Similar methodological approaches relating Carlson and colleagues' framework with disciplinary knowledge have been used by others to explore student reasoning (i.e., Scott et al., 2023). Our process involved independent coding, discussion of discrepancies, and re-defining codes until reaching a dependable consensus (Ary et al., 2014; Cascio et al., 2019; Lu & Shulman, 2008; Ryan, 1999). Table 1 shows how student descriptions of a pH titration graph would align with

A Strong Acid -Strong Base Titration

In a titration experiment, a base is added to an acidic solution at a constant rate. The graph below shows how the pH (a quantitative measure of the acidity) of the solution changes with the amount of base added. Using the graph below, explain the relationship between the pH of the solution with the amount of base added?

Graph



Explain the relationship between the pH of the solution and the amount of base added.

Fig. 2 Titration curve (chemistry task). The task included an image captured from the video showing the production of a titration curve for strong acid–strong base reaction (MrGrodski Chemistry, 2020)

Table 1 Mental actions and associated verbal behaviors when interpreting a logistic function on a titration curve

Mental actions	Description of mental actions (Carlson et al., 2002, p. 357)	Verbal behaviors in the context of interpreting a titration curve
Mental Action 1	Coordinating the value of one variable with changes in the other	Verbalizing an awareness that the pH level changes as base is added
Mental Action 2	Coordinating the direction of change of one variable with changes in the other variable	Verbalizing an awareness that the pH level increases with increasing amount of the base
Mental Action 3	Coordinating the amount of change of one variable with changes in the other variable	Verbalizing the amount of change in the pH for fixed increments of the amount of base added
Mental Action 4	Coordinating the average rate of change of the function with uniform increments of change in the input variable	Verbalizing the average rate of change of the pH with respect to successive uniform increments of the amount of the base added
Mental Action 5	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Verbalizing an awareness that while base is continuously added, the pH level increases with an increasing (instantaneous) rate of change before pH level 7 and increases with a decreasing rate after pH level 7

the five mental actions. For example, in *Mental Action 3 (MA3),* students describe the relationship between the amount of base added and the corresponding pH level. That is, they recognize that with a successive increase in the amount of base, the pH level exhibits a gradual initial rise, followed by a faster increase, and eventually a decrease in the rate of increase. In *Mental Action 5 (MA5)*, students additionally imagine how the instantaneous rate of change for pH level changes with respect to consistent change in the amount of the base. This means they are aware of the direction of concavities and inflection points on a graph. Once the team established agreement, the math expert coded all interview excerpts for mental actions. To simplify discussion, the results were split into two categories: less sophisticated or "low" covariational reasoning (engaged in Mental Actions 1–3) and more sophisticated or "high" covariational reasoning (Mental Actions 4–5).

Coding for disciplinary knowledge

The disciplinary experts from chemistry (KAB or CFB) and biology (BC or MLA) open-coded the data related to their disciplinary task, focusing on (a) the most salient disciplinary knowledge activated by a student and (b) the way(s) in which it was integrated with the student's covariational reasoning during graph interpretation. The team then defined criteria for classifying SDK as basic, mid, or high (Table 2). Basic means linking scientific terms or ideas with specific graphical regions or providing definitions with slight inaccuracies. Mid means using accurate basic-level descriptions, with qualitative explanations of how the graph shape relates to the scientific phenomenon, but the underlying mechanism or relationship between the variables that produces the shape of the graph are not expressed or understood. High means a description that is qualitatively accurate about the phenomenon and the graph shape at a mechanistic level, with reference to entities or properties or quantities that are not explicitly or immediately represented on the graph (e.g., birth rate balanced with death rate). Students may propose a thought experiment or imaginary situation based on a scientific phenomenon.

The categorization scheme was tested for reliability by means of a blind analysis. One of the chemistry authors assembled a set of paragraph-long excerpts from three interviewees whom that author believed represented the conditions of basic, mid, and high-level SDK in chemistry. These were arranged without labels side-by-side in mixed order for the other four authors to classify into any of the three categories. There was near 100% agreement in SDK ranking, except for one author categorizing someone as high level when everyone else labeled them as mid-level. Similarly, one of the biology authors repeated the process with three excerpts from interviewees for the biology task. Again, there was near 100% agreement in the blind ranking, with only one author categorizing someone as mid-level when everyone else labeled them as high. Given that the chances for this amount of agreement at random is less than 0.1%, this suggests that the interpretive scheme is trustworthy.

Integration of covariational reasoning and SDK

Having established a reliable means for categorizing levels of covariation and levels of disciplinary knowledge application, attention was turned to evidence of potential mental imagery. Evidence sought included verbalizations that referenced quantities not immediately represented on the graph, transformations of the graph (i.e., conceptual simulation; Trickett & Trafton, 2007), and verbalizations associated with iconic gestures (Trafton et al., 2006). Conceptual simulation, also referred to as "what if" reasoning, involves the spontaneous engagement of

Table 2 Examples of student responses illustrating coding scheme for scientific disciplinary knowledge

Knowledge level	Chemistry examples	Biology examples
Basic	Ivan: …. it would be based on how much, um, base is there, because like, thinking about the pH scale, … I'm not super familiar with it, …, lower numbers are more acidic, higher numbers are bases. A seven is like water, which is in like the center. Um, so it started out really low which I guess means it started out acidic. Um, and then after you've reached a certain, um, amount of that base in there, it jumped up… where this particular base falls on the pH scale, and that's why it uh, reached to that number after a certain amount of it was added to the mixture	Aubrey: Yeah, well, in between, I think that's when a population might start to realize or like kind of acknowledge that there isn't going to be quite as much or not even necessarily realized, but the, like the amount of resources becomes that they can't continue like the way that they were like, for example, um, there just suddenly like, wouldn't be enough I guess you would just say wheat, for example, to like sustain it. And I think people would subconsciously or consciously be like, all right, well maybe we don't have six kids anymore. Maybe have like three just, 'cuz it's not necessarily viable anymore to sustain, you know, eight people in a family…
Medium	Harry: So, exponential would just mean that, um, it'll start off slow and then, um, quickly accelerate which is kind of what we see in the graph is- well, there's a lot more acid than base. The acid completely controls it. Then once the base kind of cancels it out, it really quickly nears neutral, drastically increasing the pH. Then once it's neutral, it switches to really quickly increasing towards base. And then once it's all base, it kind of slows down again	Ingrid: So what I wrote was the brown tree snakes have hit carrying capacity as time goes on… …Um, but the environment can only support so many individuals based on whatever factor this species has. If they like a particular type of water, they're going to drink that water. But if you have a hundred of these, and there's only a small puddle, then if they drink that entire puddle, there's not gonna be any more water left until it fills up again after the next rainfall. Um, so all of those individuals that did not get a drink of water will die, which is what carrying capacity does. There's no- there's little or no growth after they reduce all of that, um, resource
High	Charlie:..all the H+ions are, uh, neutralized or turned into water and then it increases rapidly until it is very alkaline. So the pH is very high… Um, and then when most of the h plus ions are bound into oxygen or uh, into the water, the pH increases rapidly because the OH- concentration is increasing. … …relationship that's like the higher values don't have as much weight, I guess. I'm not quite sure. Um, so when there's a ton of, um, H+ions, um, and then you take like more away, um, until it like reaches like close to the neutral mark, which I think is seven	Mary: So that would be like population growth zero. Um, and that would just mean like for here [upper horizontal region] at least that it would be stable. So then like birth rate and death rate are equal. Um, not that it like, like the whole population isn't growing, but… …So the limit would be the carrying capacity, which is again what I'm assuming that line is. Um, but that's like the amount of resources in an area for that, um, population. So it's not that that population couldn't continue growing if they had the same conditions at that [inflection] point or even at that point [where line begins to curve upward]. Um, but that after this [inflection] point there's a, um, decrease in resources due to the size of the population…

individuals in processes such as visualizing scenarios, mentally manipulating them, and observing the resulting outcomes (Trickett & Trafton, 2007). Individuals often articulate their mental operations, including spatial transformations layered onto existing representations, when dealing with technical material (Trafton et al., 2002, 2005, 2006; Trickett & Trafton, 2007). These cognitive processes allow individuals to explore and comprehend scientific phenomena. This sense of visualization aligns with Thompson's concept that cognitive effort is required for constructing mental images of quantities (Thompson, 2011).

Gestures served as a "window" into or evidence of students' thinking (e.g., Goldin-Meadow et al., 1992) particularly those that illustrate iconic or representational expressions (McNeill, 1992) because such gestures often accompany mental spatial transformations (Trafton et al., 2006). We engaged in fine-grained discussions around the way individuals discussed the quantities depicted on the axes, how they coordinated changes between these covarying quantities, and what their language and gestures revealed about the mental imagery they constructed while engaging in reasoning about quantities and quantitative relationships.

This search for visualized thinking allowed judgment of the relative degree of integrative thinking about the graph using covariational reasoning and SDK. To establish the reliability of perceptions, another blind ranking test was done with four titration task excerpts and four biology task excerpts. One author identified four excerpts that, in our common interpretation, represented high, middle, or low covariational reasoning with science disciplinary knowledge. The set of excerpts were shuffled in sequence and presented to the other authors to rank. In each case, all authors agreed on the ranking, with one exception of an inversion of position for one interviewee. Again, the chances of this level of agreement of sequence by chance is well less than 0.1%, thus suggesting strong reliability in the interpretation regarding integration of covariational reasoning and interdisciplinary knowledge.

To support our interpretation and arguments, detailed case studies (Seawright, 2016; Seawright & Gerring, 2008) of four students are presented with contrasting levels of covariational reasoning and SDK integration through mental imagery (Table 3).

Results

Students varied in their covariational reasoning abilities and in their SDK on the population growth and titration graphical interpretation tasks (Fig. 3). On most tasks, students exhibited basic or no SDK (five out of eight students on the population growth task and ten out of fifteen students on the titration task), with most of these students also displaying lower levels of covariational reasoning on these tasks. However, a small number of these students (one in the population growth task and two in the titration task) were able to covariationally reason with little or no SDK rooted in the context of the graph. Of the students who demonstrated mid or high levels of SDK during a task, most were able to covariationally reason at higher levels. There was only one student in the population growth task and one student in the titration task who exhibited high levels of SDK but were unable to reach high levels of covariational reasoning.

Individual case analyses

Below, we present case studies of four students to showcase the different ways in which mental imagery and SDK interact (or fail to interact) as students covariationally reason. Longer student verbatim quotes are presented in figures, with perceived mental actions for covariational reasoning inserted in-line and bolded (e.g., MA2). Concurrent gestures are shown to the right, and written artifacts below. In each case study, we discuss the evidence supporting the indicated mental actions and the level of SDK achieved as well as evidence of mental imagery (or lack thereof).

Ozge understands underlying biological principles, but demonstrates no mental imagery related to changing rates

Ozge represents a rare case, demonstrating high SDK in biology, but low covariational reasoning skills. In her discussion of growth rate for the logistic growth population (Fig. 4; line B on the original task in Fig. 1), she drew on SDK when explaining the upper horizontal

Table 3 Case participants' majors, courses, tasks, and covariational reasoning (CVR) and scientific disciplinary knowledge (SDK) rankings

Participant	Major	Enrollment	Task	CVR (Mental Actions)	SDK
Ingrid	1st-year environmental science	Intro Bio	Population	High (4–5)	Mid
Ozge	1st-year wildlife conservation	Intro Bio	Population	Low (1–3)	High
Natalie	1st-year biochemistry	Intro Bio Intro Chem	Titration	High (4–5)	High
Carson	2nd-year civil engineering	Intro Chem for Engineers	Titration	High (4–5)	Absent

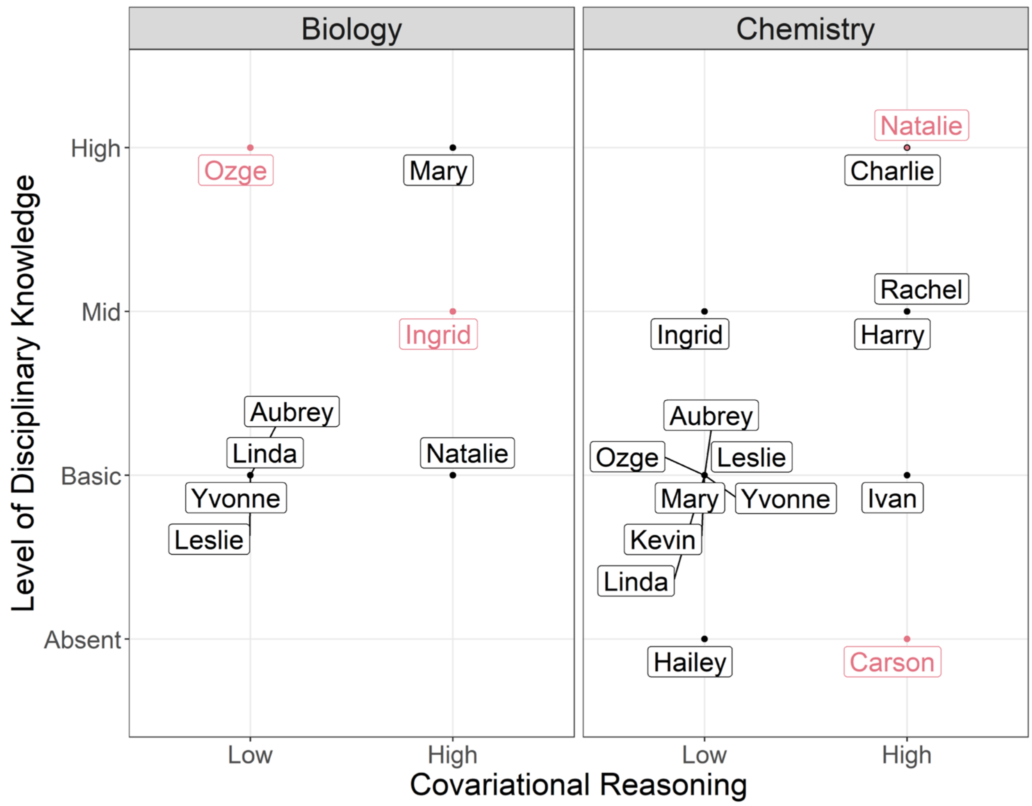


Fig. 3 Participants' representation of scientific disciplinary knowledge coincides with covariational reasoning. This figure illustrates the degree to which the utilization of scientific disciplinary knowledge (SDK) aligns with covariational thinking for 23 interview excerpts from 15 interview participants. The panels reflect which discipline the task was situated in (left: biology, right: chemistry). The y-axis represents four levels of SDK presence (basic mid, high) and absence. The x-axis categorizes students' covariational thinking levels as low (MA1–3) or high (MA4–5). Highlighted names indicate those participants discussed as individual case studies

asymptotic region. She explained (Fig. 4) that while time continues to change, the brown tree snake population has "reached the carrying capacity" so the growth rate "stays constant because that ecosystem can't fit any more [snakes]". She elaborated that this depends on the amount of resources, demonstrating basic knowledge of biology concepts by defining carrying capacity. She then elaborated on carrying capacity by explaining how the underlying birth and death rates contribute to a constant population size over time. Ozge's accurate description of carrying capacity and its relationship to both limited resources and the resulting balance between births and deaths (inputs and outputs) demonstrated she has high scientific disciplinary knowledge about this graph feature.

Figure 5 captures Ozge's initial construction of growth rate graphs for the exponential and logistic growth populations. Despite the high SDK displayed in her discussion of the region representing carrying capacity, here there is no evidence that Ozge applied biology SDK to any part of the exponential growth graph. When describing exponential growth, she fixated on a strictly numerical representation ("it") rather than using mental images of organisms: "...when it's growing exponentially, it's not growing at an exact rate, like you're multiplying it over and over again. So like the 10 times 10 would equal 1000 [sic]..." (Fig. 5). Her numerical explanation demonstrates an understanding of coordinated change of two variables—Mental Action 3: "each time it increases, it's going to increase more than the previous time" (Fig. 5). However, there is no mental image of a biological explosion of the number of organisms in relationship to time, more so a memorized unidimensional sense of multiplicative change.

Similarly, there is no evidence that Ozge applied biology SDK to any other part of the logistic growth graph. When interpreting the original logistic growth graph, Ozge did not differentiate between increasing and decreasing rates of change through the curve, and she made no mention of an inflection point. Birth rates and death rates are never mentioned in relation to these regions, and she referred only to the population as a

Speech	Gesture
So when it stabilizes - I'm also in an ecology class - so when populations stop growing	*holds right palm up in the air*
and they stay in the same place with time increase,	*in air, right hand moves to the right and taps several times*
it usually means that it's reached the carrying capacity.	*writes "x = carrying capacity"*
…And so, so carrying capacity is the number of individuals of a population that an ecosystem can support.	
So once the population growth rate gets to point x, the amount of snakes, the brown snakes in Guam stays constant	*in air, flat right hand with palm down moves to the right in a straight line*
because that ecosystem can't fit any more of them. Or if they had anymore, there would be a problem with the population.	*in air, flat hand with palm down moves quickly to the right*
And so it stays here because the amount of resources that the ecosystem has is only enough to support this amount of snakes.	*taps pen on horizontal region of line*
…there will still be more snakes born, but then the birth and death rates will be constant. So when- even though there's more births happening, like the amount of deaths is going to be the same and so- or not the same, but like usually when you get to K…	*points at horizontal region of line*
it's just like- if, if this is x, it'll go like a little bit up and down.	*draws straight line to the right of her graph* *draws an oscillating line over previously drawn line*
But overall it'll stay around the same point. Unless something happens to the ecosystem, then it will decrease again. But, even with the- if the birth rate of the snakes were to increase, the young wouldn't survive because there's not enough resources to support them. And	
And so the ecosystem only has enough resources to support this amount even if there are more snakes. And so some of them will just die.	*taps on horizontal region of line*

Fig. 4 Ozge's explanation of birth/death balance in the upper region of the logistic population curve

whole and never individual organisms. When tasked with drawing the graph of population growth rate over time for the logistic graph (which would be a derivative peaked shape), it became clear she struggled to understand the concept of rate; she drew the same exact graph shape depicted in the original population size versus time graph (Fig. 5). The interviewer prompted: "…I'm trying to understand the difference between these two graphs, population size versus time, population growth rate versus time, for the same brown tree snakes," and Ozge responded, "So I kinda just interpret it as the same." Accordingly, her references to 'population growth rate' in Figs. 4 and 5 could be replaced with 'population size' and the meaning would remain practically unchanged.

Ozge's case suggests that having high levels of disciplinary knowledge does not guarantee the ability to produce mental images that support high levels of covariational reasoning. During the interview, Ozge did not demonstrate an ability to coordinate the average or instantaneous rates of change of the function with changes in the independent variable (i.e., Mental Actions 4 or 5). Although at one point Ozge describes how birth and death rates balance at carrying capacity, which suggests she understands the average rate of change is zero (Fig. 4), she does not explain the average rate of change elsewhere on the graph. While lack of discussion from a biological perspective about other graph regions does not necessarily mean Ozge had no SDK to apply to these regions, there is a singular focus on carrying capacity and no evidence of mental imagery about the other regions of either graph.

Mental imagery helps Ingrid engage in more sophisticated covariational reasoning about changing rates

Ingrid demonstrates mid SDK and high covariational reasoning skills, using both biological and non-biological mental imagery. She first used biology mental imagery (Fig. 6) when coordinating the amount and direction of change in population size with time—Mental Action 3. She demonstrated an understanding of change in the tree snake population over the domain of a logistic function by verbalizing how the population increases initially then it "level[s] off" when it reaches the "carrying capacity" (Fig. 6). Coincident with describing this, Ingrid mimicked drawing a concave line on the graph (Fig. 6), stating that the population hasn't "dipped" or gone "extinct". Hypothetically transforming the graph to 'dip' and evaluating the result (implications for the population) suggests Ingrid was engaging in a conceptual simulation, running a mental model involving the represented quantities and mentally transforming the graph.

Ingrid continued using biology mental imagery, explaining the concept of carrying capacity, how a species' survival is tied to resource availability in the

Speech	Gesture
And so as the time increases, the population growth rate [line A] increases exponentially. **(MA3)**	
And… when it's growing exponentially, it's not growing at an exact rate, like you're multiplying it over and over again. So like the 10 times 10 would equal 1000. And then you're multiplying it more, so it'd be like 10,000 or like 100,000. So you're multiplying it more every single time that you go.	*in air, right hand circles to the right several times*
And so as time increases, the growth rate is going to be increasing.	*in air, right hand circles to the right twice*
But each time it increases, it's going to increase more than the previous time **(MA3)**. So it's not like a steady rate of growth.	
…for the brown tree snakes in Guam…. we're going to draw the same figure again, but this time for B. So as time increases, population increases until point X **(MA3)**. When population growth rate reaches point X, it stabilizes and stops increasing. And then as time increases, B remains constant after point	*sketches y and x-axes, a line with same shape as line B, and a line intersecting the y-axis at the same height as the horizontal asymptotic region labeled "x"*

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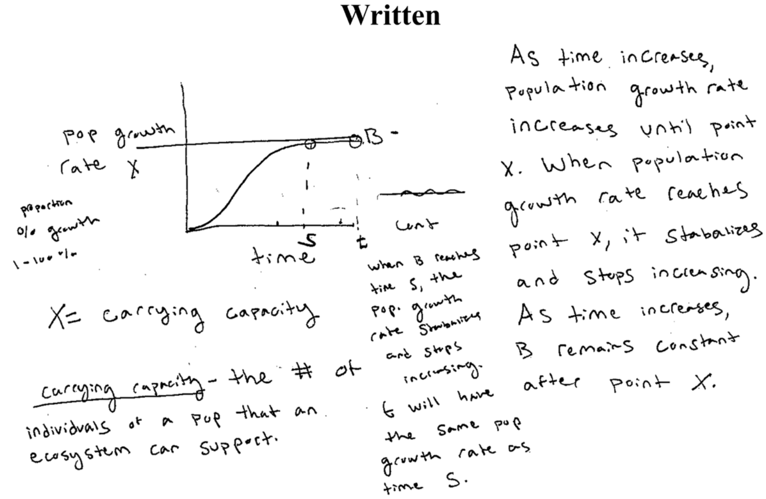

Fig. 5 Ozge's description of population growth rate, including her sketches of logistic growth rate versus time

Speech	Gesture
But, um, the population of brown tree snakes in Guam ends up leveling off. That's not a bad thing.	*hovers pen above the flattening portion of the line for brown tree snakes*
Um, because they haven't dipped down	*mimics drawing a concave down line*
and completely like been extinct.	
They're just not growing at this point in time. ...	*points to the upper horizontal flat portion of the line*
As time increases the brown tree snake population logistically grows or increases **(MA3)**.	*writes below graph as she speaks*
But in this point it's sort of just leveled off right now...	*briefly points at horizontal plateau*
So, what I wrote was the brown tree snakes have hit carrying capacity as time goes on...**(MA3)**. ...the environment can only support so many individuals based on whatever factor this species has. If they like a particular type of water, they're going to drink that water. But if you have a hundred of these, and there's only a small puddle, then if they drink that entire puddle, there's not gonna be any more water left until it fills up again after the next rainfall. Um, so all of those individuals that did not get a drink of water will die, which is what carrying capacity does. There's no- there's little or no growth after they reduce all of that, um, resource. ...	
Um, but for this population, the brown tree snake they're- they hit a point	*makes red dot at inflection point*
where they've hit the fastest possible growth they can **(MA3)**.	
And then after that point, it decreases.	*draws a green arrow from the inflection point*
And that's- I think what people in math call like a point of influx or something **(MA3)**.	

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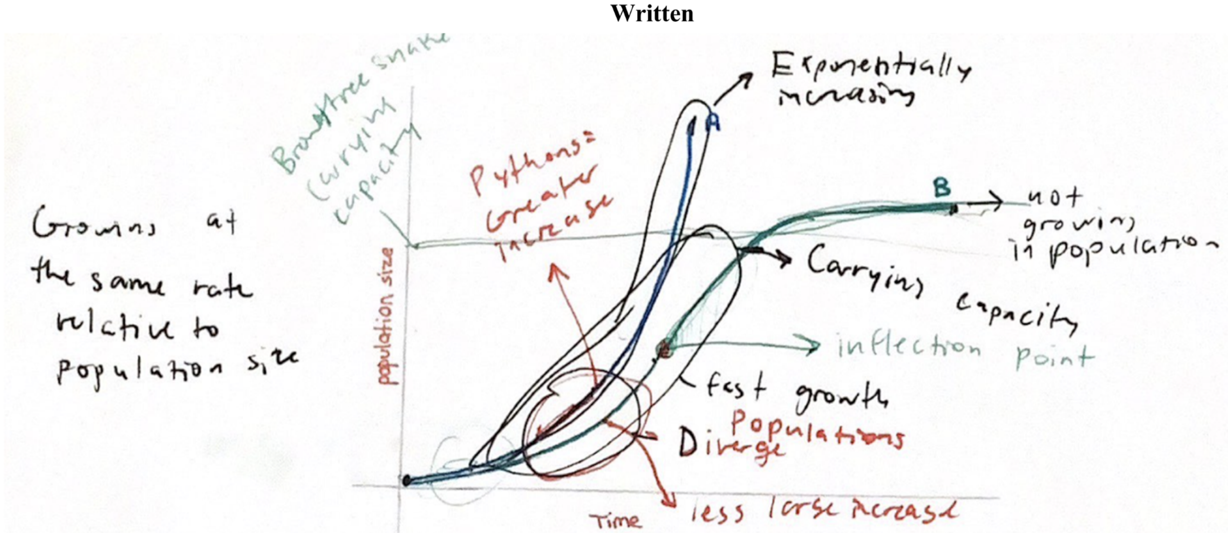

Fig. 6 Ingrid's description of brown tree snake growth at carrying capacity and across the logistic function. Below the graph Ingrid wrote: "A = Exponential; As time increases, Burmese pythons population exponentially increases. B = Logistic; As time increases the Brown tree snake population logistically increases"

environment to "only support so many individuals" (Fig. 6). She elaborated by using water as an example of a limiting resource with an imagined situation of "a hundred [individuals], and there's only a small puddle," explaining how that will result in the death of individuals and ultimately limit population growth. Ingrid's use of an imaginary situation demonstrates that she can actively construct and connect this biological concept of capacity to how the population size and population growth rate are depicted in the graph.

Ingrid's strong mental imagery became a critical aspect of her ability to construct graphs of population growth rate over time (Fig. 7). However, instead of discussing organisms in a population, she used mental imagery of pens to explain exponential growth and how logistic growth (including decay of rate) would be different (Fig. 7). Here, Ingrid has demonstrated an awareness of the instantaneous rate of change for the brown tree snake's growth. She verbalized that the population size increases with increasing rate, then increases with

Speech	Gesture
...the time interval would decrease as growth rate increases for the Burmese Python. Um, so... the length of time would become shorter. Um, as the population grows, like you would need less time to have 10 more people. Or say, I have this pen and it's magic and... it pops into two in one second,	*picks up a pen, then places it on the table*
and then it pops into four in like 0.5 seconds. and it pops into eight in like even less time. You would need less time to have double the population.... And for the brown tree snake... the time interval would increase as growth rate decreases. So the interval between time a and time b would become longer. I could... get a cup of coffee and this pen could say pop into four pens and then I could go and watch a two-hour long movie, and this pen would be eight pens, then I would wait an entire day and this pen would not even be 16 pens.... ...you would have to wait much longer for that population to grow. You could have- it's still increasing, yes, but at a rate that is very slow and then it keeps getting slower until the growth rate just levels off and stays the same **(MA5)**.	
…. So in this initial graph,	*indicates rightmost point of line B in provided graph*
you see that the population size is not increasing	*draws a line from a point on the y-axis equal in height to where the line plateaus*
which means the growth rate would be zero.	*returns pencil to her brown tree snake graph*
So actually this would go down.	*on tree snake graph, continues curve downward in pencil (Fig 7a)*
Um, so instead of having just this level off,	*indicates initial plateau (in red pen) on Fig 7a*
this is the fastest the population growth rate will be, and the growth rate will eventually hit that fastest point and then just stop growing **(MA5)**.	*draws line from y-axis to curve maximum on Fig 7b*
So, because this population	*traces upper flat region of line B on provided graph*
is not really growing you assume that growth rate is zero because there's no one being produced. Um, so growth rate would be zero here, which means there's the same amount of people or individuals, or snakes **(MA5)**.	*writes '0' on x-axis of graph (Fig 7b)*
Um, time is still increasing, it's just the growth rate is just not increasing at all **(MA5)**. It's just done.	*traces x-axis from left to right*
... Time one is the maximum growth rate. Time two	*draws line from curve to x-axis at '2'*
is- is half the maximum growth rate.	*Ingrid pauses to write, draws line from y-axis to line at '2'*
And then time three- time three is no growth.	*points at '3', Ingrid writes*
So it's, uh, it's zero of the growth rate. It's like not any percent of the growth rate **(MA5)**.	

Written

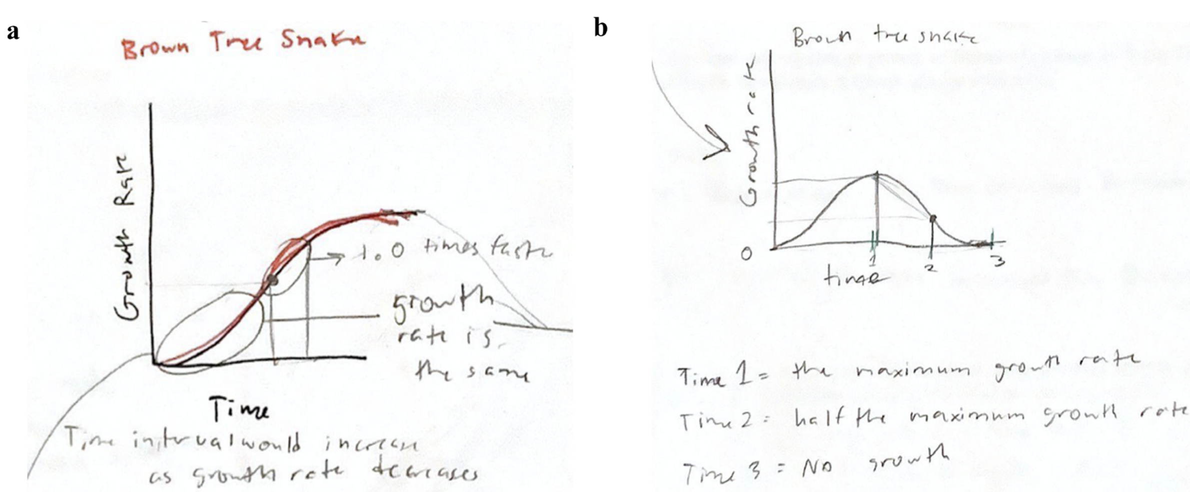

Fig. 7 Ingrid describes population growth rate versus time, constructing initial (**a**) and corrected (**b**) brown tree snake graphs. Ingrid initially re-draws the population size versus time graphs for both types of snakes, then corrects them. Below her sketches she wrote: (a) "Time interval would decrease as growth rate increases" and (b) "Time interval would increase as growth rate decrease". Below her corrected sketch for brown tree snakes (c), Ingrid wrote: "Time 1 = the maximum growth rate, Time 2 = half the maximum growth rate, and Time 3 = no growth."

decreasing rate—Mental Action 5 (Fig. 7). This illustrates Ingrid's sophisticated understanding of the covarying relationship between growth rates, time intervals, and population size.

Midway, Ingrid realized she had drawn the graph for the logistic model incorrectly. She indicated where the line plateaus on the provided graph and then looked at her own sketch (Fig. 7a) while verbalizing a spatial transformation ("this would go down") which she then carried out by extending her curve downward. Ingrid redrew the graph (Fig. 7b) after which she stated "…that makes a lot more sense." Ingrid was aware that the inflection point represents "the fastest the population growth rate will be," (Fig. 7) which she indicated corresponds to "Time 1" (Fig. 7b). When considering chunks of time, Ingrid was able to provide a detailed articulation of how the change in population size and rate is coordinated with the time it takes to change, which allowed her to construct a smooth curve for population growth rate. This indicates Ingrid can coordinate the instantaneous rate of change of population growth with the change in time across the entire function—Mental Action 5. While Ingrid's exact cognitive processes are invisible, well-articulated images both from a biological context and from a non-biological context seemed to enhance her ability to interpret the population size graph and support her construction of a smooth population growth rate curve. We interpret this to suggest that Ingrid used mental imagery to engage in sophisticated covariational reasoning on the population growth task.

Mental images of the titration process help Natalie consistently tie graph shape and physical reality together

Natalie demonstrates high SDK and high covariational reasoning when interpreting the chemical titration graph.

Figure 8 provides Natalie's initial response to the task: she described how "the base continues to get added gradually" (Fig. 8) and how it reacts with the acid already present, resulting in pH change: "…the OH's in the base that's added will react with the acid that's already in there and slowly…turn it into… water." Beyond the inflection point, Natalie described that most of the acid has been consumed and continuing to add base makes the solution more basic over time. Although her description of the chemical species present at and beyond the inflection point is partially inaccurate, Natalie was able to coordinate the change in amount of base added with the change in the pH with explicit attention to the direction of the change—Mental Action 2.

Throughout her response, Natalie associated more than one level of chemical description with the line of the graph: namely, with changing concentrations of two chemical species at the particulate level (hydroxide ions $[OH^-]$ and protons $[H^+]$), as well as with overall acidity or basicity levels as indicated by pH at the macroscopic level. This is evident, for example, in Natalie's identification of a point in the center of the graph (Fig. 8, point she marked as 'x' and labeled "Base = acid") which she explains as representing when "the concentration of… hydroxide is equal to the… protons," as well as representing "equilibrium" which she defines as "no more acid or more base" (Fig. 8) and a neutral pH of approximately seven. Her phrasing here and in Fig. 9 suggest such disciplinary knowledge supported clear mental images of the x-axis as it relates to amount of base added and the y-axis as pH.

Natalie demonstrates in Fig. 9 that she can coordinate the average rate of change of the pH with respect to uniform incremental changes of base added—Mental Action 4. This is evidenced through Natalie's articulation of distinct differences in the rate of change of the pH before and after the inflection point as uniform increments ("drops") of base are added to the solution. Natalie expressed the overall differences in rate of change of the pH by describing the horizontal asymptotic regions as "gradual" and the center region as a "spike" (Fig. 8). Elaborating, for example, on the rate of change leading into the right tail of the graph, Natalie explained how "drops of base… going in" to the solution produces "a constant increase of OH… so it [the pH] still goes up a little bit."

In explaining what causes the line on the graph to go up "a little bit", Natalie made several gestures and verbalizations suggestive of the use of mental imagery. Stating that the pH line "won't go up like that," Natalie traced a line with a slightly greater slope in air above the upper horizontal asymptotic region she is referencing. That is, Natalie appeared to engage in a conceptual simulation, wherein she refers to transforming the representation spatially in a hypothetical manner, using gestures to do so 'on top of' the existing representation. She repeated a similar simulation at the end of the excerpt, referring to a hypothetical line with an even steeper slope, once again tracing it above the titration curve. These transformations are aligned via comparison to the existing line ("…it still goes up, but it's not… really far up"; Fig. 9). In between these two hypothetical transformations, Natalie made an iconic gesture mimicking "drops of base… going in" to the titration solution, which suggests a mental image of the titration process. She associated these drops with a "constant increase of OH."

Natalie could provide a detailed description and demonstrate sophisticated covariational reasoning on the titration task, seemingly supported by disciplinary knowledge that provides clear mental images of the quantities represented on the axes and their real, physical relationship. Natalie actively recalled disciplinary ideas

Speech	**Gesture**
So, I have seen this before. So like this part at this point right here,	*points pen at the inflection point traces pen up and down center region of graph*
uh, this is when the- the, um, the concentration of, uh, hydroxide is equal to the, um, the protons, like the hydrogens.	*pen hovers above inflection point traces circle in air above inflection point*
And like here, there's no base added. So it's like all acid and the- and as the base, uh, contin- the base continues to get added gradually, the, um, the OH's in the base that's added will react with the acid that's already in there and slowly start to take it up and turn it into, um, water **(MA2)**.	*indicates leftmost point of entire curve*
This point is when it's at the equilibrium between the two so it's like about seven 'cuz it's neutral. There's no- there's no more acid or more base.	*indicates inflection point traces a horizontal line from the inflection point to the y-axis*
And then when we get to this point, it's when the, um, uh, when the acid is, um, being most- is most- has mostly been taken up	*traces from inflection point to upper horizontal region stops in the middle of the upper asymptotic region*
and you and, when you keep adding it, then there's more and more base, which makes it slightly more basic as time goes on **(MA2)**.	*circles beaker in lower right corner of representation slowly traces the upper asymptotic region*

Written

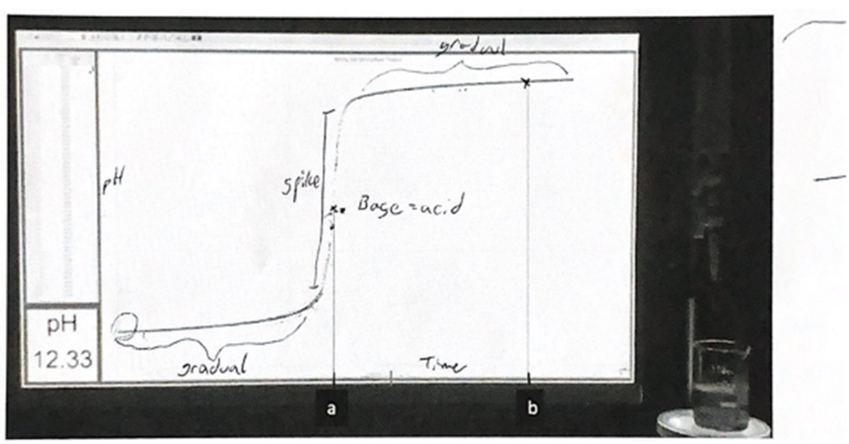

Fig. 8 Natalie describes what different regions of the titration curve represent chemically. Below the graph in response to the prompts, Natalie wrote: I think… "the pH becomes more basic as the base is added" Because… "the OH- bonds with the H+, that makes the solution acidic to become H2O which is neutral and when the majority of H+ is taken up there is excess OH- still being added making the solution basic."

Speech	Gesture
So there is still a little bit of acid, so it won't go up like that.	*in air, above the upper horizontal region, she traces a straight line with slightly greater slope*
But you're still adding, uh, the drops of base here going in.	*indicates titration set-up image on the page with little circles* *moves pen down and up once, indicating a 'drop'*
So there is still a- a constant increase of OH that-	*hovering above the graph, pen moves from left to right*
so… it still goes up a little bit,	*traces over the upper horizontal region left to right* *traces back and forth over the same region several times*
but it's not as- it- it- it- still goes up, but it's not like, like really far up **(MA4)**.	*in air, traces a straight line with a much steeper slope*

Fig. 9 Natalie describes how the rate of pH change changes with incremental additions of base

pertaining to the titration experiment, such as neutralization reactions and how addition of OH^- affects H^+ concentration and therefore pH. In the excerpts presented and throughout the task, her descriptions were explicitly and consistently grounded in the concrete context of the titration (e.g., referring to and gesturing "drops", the interactions and reaction between the chemical species). Even with some inaccuracies from a chemical standpoint (e.g., incorrect use of the word equilibrium; inconsistent discussion of when acid analyte is depleted), the use of disciplinary knowledge helped Natalie create mental images of how the x- (amount of base) and y- (pH level) variables covary throughout the graph. Although Natalie did not explicitly mention abstract mathematical concepts, it is evident that she implicitly utilizes some. For example, she did not explicitly say "point of inflection", but her recognition of that point and its influence on the rate of change indicates an understanding of inflection points and the curvature they produce on the graph.

Carson uses mathematical mental imagery to engage in sophisticated reasoning, but does not connect to the physical context

Carson stands out because he evinces no scientific disciplinary knowledge while demonstrating high-level covariational reasoning skills. He recognized points of inflection and conceived of the graph as a representation of simultaneously covarying quantitative relationships (MA5), but there was a disconnect when it came to relating this to the titration experiment and chemistry concepts. In Fig. 10, Carson demonstrated a sophisticated understanding of instantaneous rates of change.

He noted that "every drop… changes the pH… significantly"—Mental Action 4—and articulated that the rate of pH change before and after the inflection point are different, with "[the rate of change]… getting faster" before and "[pH] increas[ing] at a decreasing rate" after—Mental Action 5. Through this, Carson related the instantaneous rate of change for pH to incremental change in the amount of the base ("every drop") and showed his awareness of the direction of concavities and inflection points on the graph, demonstrating sophisticated covariational reasoning skills.

Yet, at no point in the interview did he spontaneously invoke any chemical concepts (e.g., reaction, concentration, or acidity) or explain the titration process. When prompted by the interviewer to clarify the concept of the inflection point in the context of pH and the amount of base added, Carson explained how the pH, a quantity on the y-axis, and the amount of the base, on the x-axis, form a logistic graph where "…the pH changes more and more with every drop up until that [inflection] point and the pH changes less and less with every drop as it keeps going" (MA5). Although this is a sophisticated explanation of how the graph illustrates the covarying quantitative relationship between "every drop" and pH change, one could easily replace 'pH' with 'y-variable' and his meaning would not change. That is, the variables forming the graph are treated as abstract entities rather than being conceptualized within a chemical process. Carson's only direct reference to his chemistry knowledge is recollection of performing titration experiments in chemistry classes, where he admits "…it's hard to describe the mathematical relationship in words, like, between the specific, um, like a relationship of the base and the pH."

Carson's response illustrates that mental imagery can be formed within mathematical thinking. However, it also demonstrates that sophisticated covariational reasoning with a scientific graph is not a guarantee that scientific sense-making will also be engaged.

Discussion

This study explored the role of scientific disciplinary knowledge (biology and chemistry) in students' covariational reasoning during graph interpretation: the extent to which students engaged SDK (RQ1) while interpreting the graphs, and the ways in which mental imagery, primarily derived from SDK, contributed to their covariational reasoning (RQ2).

Speech	Gesture
Uh, at that point it's just like the- every drop changes it- changes the pH a lot more significantly than it does either before that point	*right hand, perpendicular to table, moves left slightly*
or after it like starts to slow down again **(MA4)**…	*same hand moves right slightly*
Here, it [the rate of change] was getting faster and faster until it got to a.	*points at lower region where line dramatically curves* *traces over lower curve and line immediately before 'a'* *points at 'a'*
And then there's the point of inflection there. And then it turned to a negative	*traces line immediately after 'a'*
rate of change because it started slowing down	*re-traces line immediately after 'a'*
until it got to	*traces upper horizontal asymptotic region*
like-	*flat right hand, perpendicular to the table, moves from left to right several centimeters*
by the end, it was very slow again…. It's still- it still increases, but it increases at a decreasing rate **(MA5)**.	*pointing at curve after point "a"*

Written

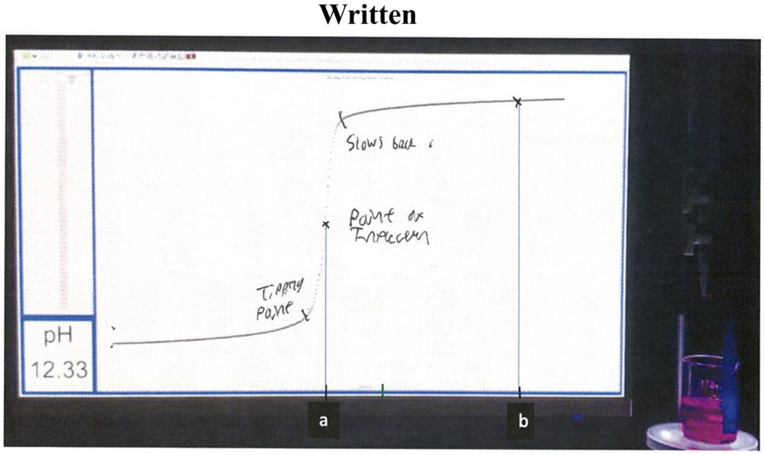

Fig. 10 Carson describes rate of change along the entirety of the titration curve

With respect to the first research question, Fig. 3 shows the extent to which scientific disciplinary knowledge usage is coincident with covariational reasoning. By dichotomizing each axis into two levels ("High" for high and mid levels; and "Low" for basic or nil levels), one can see that, irrespective of discipline, 12 student excerpts were classified as low on both types of thinking and six were high on both. Only five interviews were mixed (high on one and low on the other), nearly evenly split between high/low and low/high. Though not with sufficient power for reliable statistical inference, the trend suggests that higher functionality with SDK goes along with higher demonstrated covariational reasoning ability. This 2×2 simplification of Fig. 3 will be called the SDK/COV matrix for discussion purposes.

In the lower left quadrant of this matrix, students who demonstrated low SDK also tended to demonstrate lower levels of covariational reasoning. These students did not express biological or chemical concepts to explain the underlying phenomena driving the shape of the graph, and they did not explain the rate of change depicted in the scientific graphs, either in terms of average rate of change (MA4) or instantaneous rate of change (MA5). Many of the interviewed students are in their first year and have not yet gained enough experience with disciplinary concepts, practices, or language, which is evidenced in the limited ability to describe and explain these graphs in scientific or mathematical terms.

In the upper left and lower right quadrants, students exhibited more facility in one characteristic but not the other. Three students in the lower right quadrant demonstrated little to no SDK, while discussing rates of change satisfactorily as they reasoned through the graph shape. Conversely, two students in the upper left quadrant exhibited high SDK, but it was not accompanied by sophisticated explanation for rates of change in the graphs. It seems then that the development of scientific disciplinary knowledge and development of covariation reasoning skill do not necessarily go hand in hand, at least as evidenced by student ability to make sense of these graphical representations. Other work has found that covariational reasoning and knowledge of the relevant context can develop individually (e.g., Scott et al., 2023; Sterman & Sweeney, 2007).

In the fourth quadrant, individual case descriptions provide deeper insight—where students exhibited higher levels of both SDK and covariational reasoning. If mental imagery is critical for sophisticated covariational reasoning, as claimed by Thompson (1994), then it is valuable to seek evidence for that with the student interviews in this quadrant. This is in fact what was found. Natalie and Ingrid both exhibit mental imagery related to their science disciplinary knowledge. Ingrid notably used the mental image of pens doubling over time more quickly or more slowly as an analogic substitute for animal population changes. Natalie used quantitative comparisons at the macroscopic level (acidity and basicity) and particulate level (concentrations or numbers of hydrogen ions and hydroxide ions) to relate amount of base added to corresponding changes and rates of change of pH. The students here are using concrete quantitative representations grounded in the scientific discipline: quantities of physical entities and the rates of changes in these quantities.

As a contrasting case, Ozge demonstrates high SDK but not high covariational reasoning. She is able to correctly explain how, due to resource limitations, birth and death rates balance to explain carrying capacity in the logistic growth graph. However, she shows no evidence of using mental images to coordinate changes in population size over time on any other part of the logistic growth graph.

Carson provides an alternative contrasting case. He exhibits no SDK but is able to explain instantaneous rate of change, the highest level of covariational reasoning, between amount of base added and pH across the entirety of a logistic function representing a titration curve. While he used the language of the axes' labels to describe incremental changes in the quantities, their direction of change, their rates of change, and the changes in those rates, he relied solely on mathematical thinking. He never connected to chemical concepts that are essential for making sense of the phenomenon represented by the graph or for making inferences about the chemistry there. Curcio (1987) would describe this as not being in a position to "read beyond the data".

Taken together, these findings suggest the recall of prior disciplinary knowledge can form a foundation for developing mental images of quantities and quantitative relationships presented in a graph during the interpretation process, enhancing individuals' capacity to effectively interpret graphs and engage in more sophisticated covariational reasoning. Greater familiarity with the content being presented graphically accompanies clearer description, more effective interpretation, and deeper evaluation of quantitative data (Shah, 2002; Shah & Hoeffner, 2002; Shah et al., 2004). Other research has shown disciplinary knowledge is valuable when engaging with external representations, including graphs, as it supports the development of a mental model of the data which can be mentally modified to test interpretations and inferences (e.g., Chinn & Brewer, 2001; Trafton et al., 2006). Experienced scientists employ many experience-based, domain-specific resources to help them construct meaning from graphs (Bowen et al., 1999; Roth, 2012). These scientists have established a one-to-one correspondence that makes talk about the graph and the represented phenomenon indistinguishable (Roth, 2003). These previous studies have highlighted the connection between disciplinary knowledge and graphical interpretation, and this work builds on these studies by specifically exploring the mental imagery derived from scientific disciplinary knowledge and covariational reasoning in graph interpretation.

Greater familiarity with relevant disciplinary knowledge not only supports reading "beyond the data" (Curcio, 1987) to make appropriate inferences, but it may also reduce the intrinsic cognitive load experienced by an individual when engaging in a complex task like graph interpretation (Sweller, 2011; Van Merrienboer & Sweller, 2005). In interpreting representations, students must simultaneously engage disciplinary content knowledge and features of the representation itself (e.g., its conventions, labels) as they reason (Schönborn & Anderson, 2009; Schönborn et al., 2002). Familiarity ostensibly frees up mental capacity for more sophisticated reasoning.

Limitations

One potential limitation is that a math education expert conducted all interviews. This was purposeful in order to ensure that the focus would be on revealing covariational reasoning and for consistency. In contrast, science experts were not included as interviewers because the goal was to allow scientific disciplinary knowledge to emerge as prior knowledge rather than to be exhaustively excavated. Further, having multiple interviewers could have been more intimidating for students and might have constricted their responses. Thus, it is possible that opportunities to unpack student scientific understanding more fully were missed. To mitigate this limitation, the interview protocol was reviewed and modified by the biology and chemistry experts.

While the small number of participants and limited representation contexts (population biology and chemical titration) reported here do not allow for broad generalization or characterization of reasoning patterns, this study affords a deep examination of individuals' mental imagery associated with covariational reasoning. This approach lets us focus explicitly on the mental images constructed by students as part of understanding depicted scientific quantities and coordinated changes among them, which underlies covariational reasoning (Thompson, 1993). Studies with additional participants and in additional contexts are necessary to further understand the interconnection of SDK and covariational reasoning.

Conclusion

The cases examined in this study provide empirical evidence that when students construct mental images of the quantities depicted by graphs and form images of coordinated change among these quantities, they produce evidence of more comprehensive understanding that incorporates shape (function) and context (phenomenon). Recall of relevant disciplinary knowledge to construct mental images of the quantifiable attributes of the represented entities or systems helps students move beyond fixating on the overall shape of the graph to grasp the nuanced relationships represented: simultaneous changes in variables, rate of change, and how these elements relate to the graphical forms depicted (e.g., line, concave up, or concave down). For example, in chemistry, students can benefit from recognizing that a titration curve represents a covariational relationship between titrant added (amount of acid or base) and pH (reflecting amount of hydronium ions) which evolves over the course of a titration. This can support interpreting the titration curve as a representation of a dynamic chemical process rather than a static entity. In biology, focusing on discerning and coordinating changes in population size over time can help students gain deeper insight into how the balance of birth and death rates gives rise to population dynamics, and how the availability of resources influences population size, particularly in relation to carrying capacity. Encouraging students to grasp what quantities are depicted on a graph, connecting to context and to concepts, is a crucial first step to support the construction of mental images that will help them move beyond surface-level interpretation toward more meaningful interpretation of depicted relationships.

Supplementary Information

The online version contains supplementary material available at https://doi.org/10.1186/s40594-024-00492-5.

Additional file 1.

Acknowledgements
We would like to extend our heartfelt appreciation to our student participants.

Author contributions
NA, with input from MLA and CFB, designed the interview tasks and was the major data collector. NA, KAB, BC, CFB and MLA analyzed and interpreted data. NA, KAB, BC, and MLA contributed to writing the results relevant to their disciplinary expertise. NA and KAB were the major contributors in writing the manuscript. CFB edited significant portions of the manuscript. All authors read and approved the final manuscript.

Funding
The study was supported by the National Science Foundation (NSF) under Grant Number 2013427. Any opinions, findings, and conclusions or recommendations in this article do not necessarily reflect the views of the NSF.

Availability of data and materials
The interview protocols used in this study are available in the Supplementary Information. Data from interview transcripts supporting the conclusions of the article are included in part within the article and available in whole from the corresponding author on reasonable request.

Declarations

Competing interests
The authors declare that they have no competing interests.

Received: 10 November 2023 Accepted: 8 July 2024
Published online: 15 July 2024

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